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RELATIVISTIC CHARGE DYNAMICS IN ELECTROMAGNETIC FIELDS

by

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Author's Declaration

At no time during the registration for the degree of Doctor of Philosophy has the author been registered for any other University award without prior agreement of the Doctoral College Quality Sub-Committee.

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Relativistic Charge Dynamics in Electromagnetic Fields
Lauren Elizabeth Ansell

Abstract

In this thesis we consider the motion of a charged particle in strong electromagnetic background fields, having in mind applications to state-of-the-art high-power lasers. To find solutions of the Lorentz force equation of motion, we make use of Noether's theorem to identify conserved quantities of the charge dynamics. We will explain how, given enough symmetries, a dynamical system becomes integrable or, with a maximum number of conserved quantities, maximally superintegrable. Beginning with charged particles in vector background fields, we shall show that the relevant symmetry group is the Poincaré group. The dynamics for constant and univariate fields is classified, and their integrability properties are clarified. We then move on to the problem of a particle in a scalar background field and show that the symmetry group is extended to the conformal group. We then present examples of fields which include Poincaré, dilation and special conformal symmetries leading to varying extents of integrability.

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Chapter 1

Introduction

The Nobel Prize in physics this year (2018) was awarded for "groundbreaking inventions in the field of laser physics" [1]. The prize was jointly awarded to Gérard Mourou and Donna Strickland for their work in chirped pulse amplification (CPA) [2]. This process creates ultrashort high-intensity laser pulses by stretching the laser pulse to reduce the peak power, amplifying them and finally compressing the pulse. The procedure increases the intensity of the pulse as more light gets packed together as the pulse gets compressed in time. Figure 1.1 below shows the method of CPA.

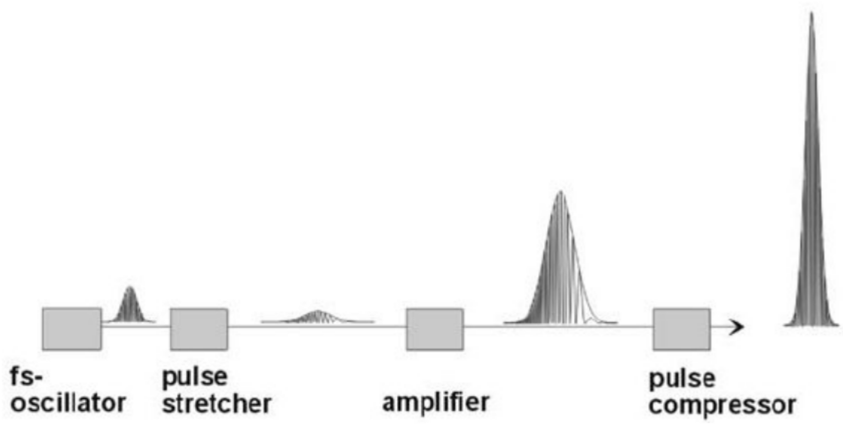


Figure 1.1: The method of chirped pulse amplification. [3]

Before the introduction of CPA, the peak power of lasers was limited to intensities in the range of terawatts per square centimeter as peak powers greater than this damaged the gain medium through nonlinear processes such as self-focusing. This self-focusing leads to plasma formation, entailing reduced beam quality. There is also the possibility of back-reflection which could damage the lasers components. Using CPA, table-top amplifiers are now able to generate pulses with peak powers of several terawatts and, in larger facilities, ultrashort pulses can reach powers of the order of petawatts. This allows for experiments to reach the relativistic regime, where the velocities involved approach the speed of light and intensities are larger than 10^{19} W/cm^2. Figure 1.2 below shows the increase in focus intensity of lasers since the creation of the first laser in 1960 to the present day.

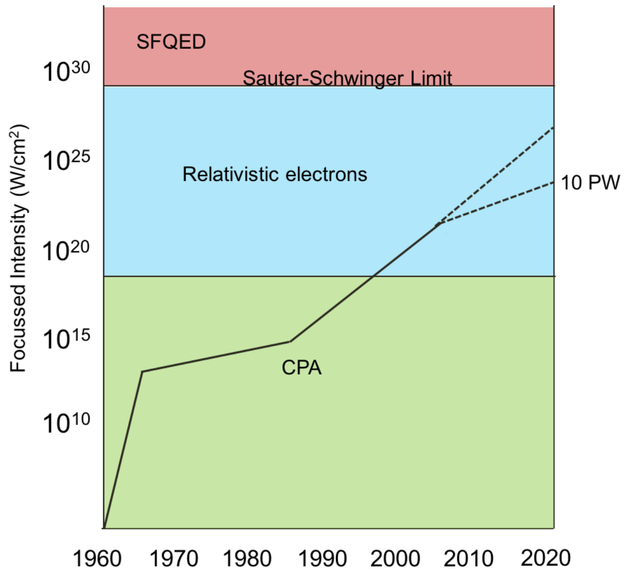


Figure 1.2: Development of laser intensity (adapted from [4]). CPA denotes the breakthrough provided by chirped pulse amplification.

New facilities across the globe are undertaking projects to take laser powers to increased intensities. At the Central Laser Facility, the Vulcan 2020 project plans to upgrade the current one petawatt (PW) laser to a short pulse beamline that will have a power of 20 PW [5]. The Nuclear Physics Facility pillar of the European Extreme Light Infrastructure (ELI-NP) [6] will involve two 10 PW ultra-short pulse lasers. These higher intensity beams will be able to produce stronger electric fields which will, among other things, allow for the creation of coherent x-ray sources. The other two pillars are the Attosecond Light Pulse Source (ELI-ALPS) which will provide light in the terahertz and x-ray frequency range in ultrashort pulses with high repetition rate and the ELI-Beamlines, a high-energy beam facility for the development and use of ultra-short pulses of high-energy particles and radiation [7].

The typical field magnitudes reached by current facilities are shown in the table below:

Power	$P \gtrsim 10^{15}\,\mathrm{W} \equiv 1\,\mathrm{PW}$
Intensity	$I \gtrsim 10^{22}\,\mathrm{W/cm^2}$
Electric field	$E \gtrsim 10^{14}\,\mathrm{V/m}$
Magnetic field	$B \gtrsim 10^{10}\,\mathrm{G} \equiv 10^{6}\,\mathrm{T}$

Table 1.1: Typical laser parameters

Laser fields are typically modelled as alternating, pulsed and null fields. A null field is one where the invariants $\mathcal{S} = (E^2 - B^2)/2$ and $\mathcal{P} = \mathbf{E} \cdot \mathbf{B}$ are zero, for E and B refer to Table 1.1. A typical example of this would be the plane wave. We assume that the laser is a classical electromagnetic background field modelled as a plane wave. Our aim is to use a realistic model of a laser field to solve for the charge dynamics. In addition to this, one can also solve for the quantum charge dynamics and Compton scattering.

To study these intense fields, we use a probe created from charged particles. The probes have two invariant quantities, the laser frequency and the laser energy density seen by the probe. From these quantities, we define two invariant parameters, a quantum energy parameter b_0,

$$b_0 = K.p/mc^2 = \hbar\omega_L/mc^2 \tag{1.1}$$

where K is the momentum of the laser, p is the probe momentum, $\hbar$ denotes Plancks constant, ω_L is the laser frequency, m is the mass and c represents the speed of light, and a classical intensity parameter a_0,

$$a_0 = (p_\mu T^{\mu\nu} p_\nu)/m^4 = Ee\lambda_L/mc^2\,, \tag{1.2}$$

where $T^{\mu\nu}$ is the energy momentum tensor, E is the electric field strength, e denotes the charge of the particle and λ_L is the reduced wave length of the laser.

Figure 1.3 below is a plot of energy versus intensity (b_0 and a_0), the red line representing the boundary between the classical and quantum regime where the “quantum efficiency parameter” $\chi = a_0 b_0 = 1$.

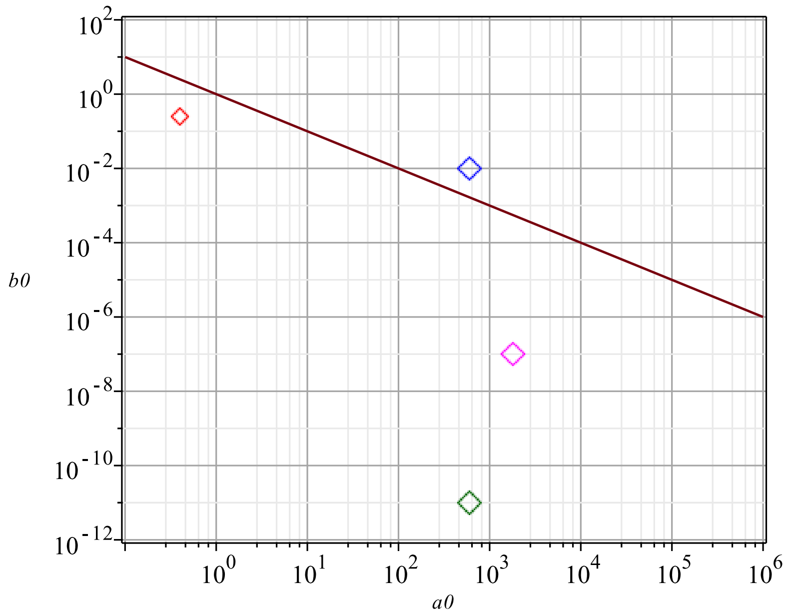


Figure 1.3: Plot of the quantum energy parameter b_0 against the classical intensity parameter a_0 with $\chi = 1$ shown by the red line. Diamonds denote the χ values achieved by experiments, red: SLAC E-144 (Stanford), blue: HIBEF II (at European XFEL), purple: Station of Extreme Light (SEL, Shanghai) and green: 10 PW, optical frequencies only.

An alternative probe of the laser field can be light itself. However, this is a quantum process and will not be the focus of this work.

In this thesis will be investigating interactions in the classical regime, at high intensity, $a_0 > 1$, but at low energies, $b_0 \ll 1$. We will show, for both vector and scalar background fields, that a symmetry of the external field implies a conserved quantity of the particle motion. Our method to then find solutions of the equation of motion will be by way of symmetry classification of the background field. We will show that for an external field with sufficiently many symmetries, we can solve for the particle dynamics exactly. To find conserved quantities and solve the equations of motion we will be using the Hamiltonian formulation. We will begin with studying particles moving in vector background fields, starting with fields that are constant in space and time before looking at examples that depend on a single spacetime coordinate. We will investigate the integrability of each case and solve the equation of motion. We will then move on to investigate the integrability of particles moving in a scalar background field and show that we now have an enlarged symmetry group, the conformal group, and present examples that possess symmetries from the conformal group and solve the resulting equation of motion.

Chapter 2

Particle Motion, Symmetry and Integrability

In this chapter we will discuss the free relativistic point particle and the symmetries it possesses. We shall also review Noether's first theorem which relates symmetries and conservation laws. We will also be considering the Hamiltonian formulations that we will be using and introduce the ideas of integrability and superintegrability. A brief outline of the conventions adopted in this thesis are as follows; the speed of light, c, will be set to one, proper time will be denoted as τ and the Minkowski metric will be represented by η and has the trace $(1, -1, -1, -1)$.

2.1 Relativistic point particle

We shall be considering a relativistic point particle of mass m and charge e. Our goal is to find the particle trajectory given by the 4-position, $x^\mu = x^\mu(\lambda)$, in Minkowski space. The parameter λ denotes the curve parameter such that the tangent vector along the curve is $\dot{x} := dx/d\lambda$. A preferred parametrisation is given by arc length s, defined through the Lorentz invariant $ds^2 \equiv c^2 d\tau^2 = dx_\mu dx^\mu$, where we have introduced proper time τ.

The dynamics of the particle is described by the action, S, which is a functional of the trajectory $x(\lambda)$ that is proportional to its arc length,

$$S \equiv S[x(\lambda)] = -mc \int ds = -mc \int d\lambda \sqrt{\dot{x}^2} =: \int d\lambda \, L(\dot{x}) \, . \tag{2.1}$$

The action is thus the integral of a Lagrangian, $L(\dot{x}) = -mc\sqrt{\dot{x}^2}$. Note that parametrisation with proper time τ implies constraint on velocities,

$$\dot{x}^2 = c^2 \, , \tag{2.2}$$

which leads to $L = -mc^2$ with obvious dimensions of energy. As usual, the equations of motion are obtained through the principle of least action, $\delta S = 0$. Following [8] we perform the variation of (2.1) immediately specialising to $\lambda = \tau$,

$$\delta S = -mc^2 \int_{x_1(\tau_1)}^{x_2(\tau_2)} \delta(d\tau) = 0 \, , \tag{2.3}$$

where the trajectory is assumed to connect fixed points $x_1(\tau_1)$ and $x_2(\tau_2)$.

The boundary variations are thus assumed to vanish,

$$\delta x_1 = \delta x_2 = 0 \;, \tag{2.4}$$

see Fig 2.1.

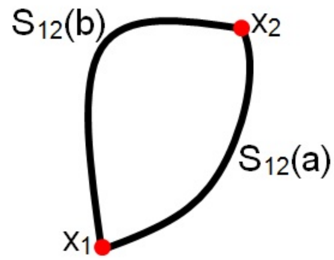


Figure 2.1: Two possible trajectories for a particle starting at the point x_1 and ending at x_2 and the associated actions $S_{12}(a)$ and $S_{12}(b)$ governing the trajectories.

When calculating $\delta(d\tau)$, we note that $cd\tau = \sqrt{dx^\mu dx_\mu}$, so that we can write,

$$\delta S = -m \int_{x_1(\tau_1)}^{x_2(\tau_2)} \frac{dx^\mu \, \delta dx_\mu}{d\tau} = -m \int_{x_1(\tau_1)}^{x_2(\tau_2)} \dot{x}^\mu \, \delta dx_\mu \tag{2.5}$$

To gain the equations of motion, δx^μ must multiply an object under the integral, which must then vanish. As derivatives are still acting on δx^μ, we rewrite the integrand as the linear combination of a total derivative and additional terms in which δx^μ appears multiplicatively. This is achieved using integration by parts and, defining the dot product between two vectors as

$a_\mu b^\mu = a.b$, we obtain

$$\delta S = -m \int_{x_1(\tau_1)}^{x_2(\tau_2)} d\tau \frac{d}{d\tau}(\dot{x}.\delta x) + m \int_{x_1(\tau_1)}^{x_2(\tau_2)} d\tau \, \ddot{x}.\delta x \, . \tag{2.6}$$

The first term is a surface term and vanishes due to (2.4), so that (2.6) reduces to

$$\delta S = m \int_{x_1(\tau_1)}^{x_2(\tau_2)} d\tau \, \ddot{x}.\delta x = 0 \, . \tag{2.7}$$

For the variation to vanish, everything multiplying δx^μ must vanish, which gives the equation of motion in its simplest form as

$$\ddot{x} \equiv \dot{u} = 0 \; . \tag{2.8}$$

The absence of 4-acceleration immediately shows that a free particle moves with constant 4-velocity,

$$u^\mu := \frac{dx^\mu}{d\tau} \equiv \dot{x}^\mu = u_0^\mu = const \; . \tag{2.9}$$

A second integration yields the trajectory,

$$x^\mu(\tau) = x_0^\mu + u_0^\mu \tau \; , \tag{2.10}$$

which corresponds to the familiar uniform linear motion of a free particle starting at initial position x_0. Henceforth, we choose natural units and set $c = 1$ for simplicity.

2.2 Symmetry

The word symmetry derives from the Greek words 'sym', meaning 'with' and 'metron' meaning measure. Symmetry occurs commonly in nature such as the reflection symmetry of a butterfly's wings and the discrete rotational symmetry of snowflakes. In physics "a symmetry or symmetry transformation of a geometric object in Euclidean space is an isometry which maps the object onto itself" [9]. Symmetry is a form of invariance, which is a mathematical property held by objects that remain unchanged when certain transformations are applied to them. Symmetries can be classified as either continuous or discrete. Examples of discrete symmetries include space and time reversal (P and T) or charge conjugation symmetry, C. Continuous symmetries of space-time correspond to invariance under translations and rotations (homogeneity and isotropy of space-time). The associated symmetry transformations are combined into Poincaré transformations which map

$$x^\mu \to x'^\mu = a^\mu + \Lambda^{\mu\nu} x_\nu \ . \tag{2.11}$$

where Λ represents a finite Lorentz transformation. If proper orthochronous (no space-time reflection), it is continuously connected to identity. Therefore it makes sense to consider infinitesimal transformations *close* to the identity by writing $\Lambda x = x + \omega x$,

$$x'^\mu - x^\mu \equiv \delta x^\mu = (a^\mu + \omega^{\mu\nu} x_\nu)\, \delta\epsilon \ , \quad \omega^{\mu\nu} = -\omega^{\nu\mu} \ . \tag{2.12}$$

Thus, a Poincaré transformation depends continuously on the 10 parameters a^μ and $\omega^{\mu\nu}$ representing four translations and six Lorentz transformations. It is straightforward to show that the transformations form a group.

In the case of a particle trajectory, a symmetry will map a trajectory to a trajectory,

$$x(\tau) \to x'(\tau)\,, \tag{2.13}$$

which by the principle of least action implies that the action remains unchanged,

$$S[x] = S[x']\,. \tag{2.14}$$

The Lagrangian of the free particle depends only on the velocity $\dot{x}$ with $\dot{x}' = \Lambda\dot{x}$ which is pure Lorentz and is therefore clearly invariant under translations, for which $\dot{x}' = \dot{x}$. Also we have constructed $ds = (dx.dx)^{1/2}$ so that it is also Lorentz invariant. Therefore it is clear that the free particle has full Poincaré symmetry.

2.3 Noether's First Theorem

This year marks the 100th anniversary of Noether's theorem, a key element in the formulation of new theories and the basis of the standard model of physics. Noether's first theorem states that "every differential symmetry of the action of a physical system has a corresponding conservation law" [10], and also that the converse, in particular circumstances, is true, that having

a conservation law implies a corresponding symmetry. In the case where we have topological conserved quantities, these do not always have a corresponding symmetry.
Noether's second theorem also concerns symmetries and it relates the symmetries of an action functional with a system of differential equations. This theorem is often used in gauge theories, which are the basis of all modern field theories.

For a symmetry transformation the boundary variations are nonzero, $\delta x_1 \neq 0$ and $\delta x_2 \neq 0$, see Fig 2.2.

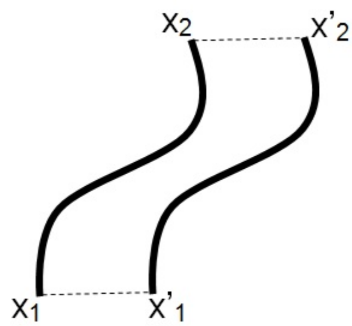


Figure 2.2: The original trajectory from points x_1 to x_2 translated to the new points x_1' and x_2'.

Thus, from (2.6), the variation of action is reduced to a surface term,

$$\delta S = -m \int_{x_1'(\tau_1)}^{x_2'(\tau_2)} d\tau \, \frac{d}{d\tau} \, \dot{x}.\delta x + m \int_{x_1'(\tau_1)}^{x_2'(\tau_2)} d\tau \, \ddot{x}.\delta x = -m \int_{x_1'(\tau_1)}^{x_2'(\tau_2)} d(\dot{x}.\delta x) \tag{2.15}$$

as the equation of motion term vanishes. Hence

$$\delta S = -m\left(\dot{x}_2.\delta x_2 - \dot{x}_1.\delta x_1\right) . \tag{2.16}$$

Setting $\tau_1 = \tau$ and $\tau_2 = \tau + d\tau$, this becomes the infinitesimal statement

$$\dot{x}.\delta x = \text{const} . \tag{2.17}$$

We now rewrite this in terms of the Lagrangian,

$$-\frac{\partial L}{\partial \dot{x}^\mu}\,\delta x^\mu =: p_\mu \delta x^\mu = \text{const} , \tag{2.18}$$

where $p_\mu = m\dot{x}^\mu$ denotes the *canonical momentum* for the free particle. We note that the mass-shell constraint follows from (2.2),

$$p^2 = m^2 . \tag{2.19}$$

A general symmetry transformation is written as

$$\delta x^\mu = \xi_i^\mu \,\delta\epsilon^i , \tag{2.20}$$

where the index i labels the infinitesimal parameters. The advantage of identifying conserved quantities is that it leads to the system becoming easier to solve as they impose constraints on the particle motion, see below. We define a conserved *Noether charge* for each parameter, Q_i, as

$$Q_i := \frac{\partial L}{\partial \dot{x}^\mu} \, \xi_i^\mu = p \, \xi_i \, , \tag{2.21}$$

it then follows that a constructive way to *find* Poincaré symmetry is to calculate the time derivative of $Q = \dot{x} \, \xi$:

$$\dot{Q} = \frac{d}{d\tau} (\dot{x} \, \xi) = \ddot{x} \, \xi + \dot{x} \, \dot{\xi} = 0 \; . \tag{2.22}$$

Using the equation of motion, $d/d\tau = \dot{x}.\partial$ and symmetrisation this becomes

$$\dot{Q} = \dot{x}^\mu \dot{x}^\nu \partial_\nu \xi_\mu = \frac{1}{2} \dot{x}^\mu \dot{x}^\nu (\partial_\mu \xi_\nu + \partial_\nu \xi_\mu) = 0 \; . \tag{2.23}$$

For this to vanish we require

$$\partial_\mu \xi_\nu + \partial_\nu \xi_\mu = 0 \, . \tag{2.24}$$

This is the Killing equation for flat Minkowski space [11] and has the 10 parameter solution

$$\xi^\mu = a^\mu + \omega^{\mu\nu} x_\nu \, , \tag{2.25}$$

which we recognise as the infinitesimal Poincaré transformation from (2.12). So the Poincaré Noether charges are collected into

$$Q = p.\xi = p.a + \frac{1}{2} \omega^{\mu\nu} (x_\mu p_\nu - x_\nu p_\nu) \equiv p_\mu a^\mu + \frac{1}{2} \omega^{\mu\nu} M_{\mu\nu} \; . \tag{2.26}$$

We can then expand a^μ and $\omega^{\mu\nu}$ in terms of basis vectors and this results in the 10 Noether charges, p^μ and $M^{\mu\nu}$ where $M^{\mu\nu} = x^\mu p^\nu - x^\nu p^\mu$. The spacetime translations are generated by p^μ, and $M^{\mu\nu}$ generate the Lorentz transformations. These obey the Poincaré algebra, which, if we think of p^μ and $M^{\mu\nu}$ as operators, are given by the commutation relations as follows,

$$\begin{aligned} [p^\mu, p^\nu] &= 0 \\ \frac{1}{i}[M^{\mu\nu}, p^\rho] &= \eta^{\mu\rho} p^\nu - \eta^{\nu\rho} p^\mu \\ \frac{1}{i}[M^{\mu\nu}, M^{\rho\sigma}] &= \eta^{\mu\rho} M^{\nu\sigma} - \eta^{\mu\sigma} M^{\nu\rho} - \eta^{\nu\rho} M^{\mu\sigma} + \eta^{\nu\sigma} M^{\mu\rho} \end{aligned} \tag{2.27}$$

where η is the Minkowski metric.

2.4 Hamiltonian Formulation

The equations of motion (2.8) are a set of second order differential equations. However, it is possible to express these instead as a system of first order differential equations. To do this we introduce the Hamiltonian formulation. To obtain the Hamiltonian, or Hamiltonian function, we try a covariant Legendre transform of the Lagrangian,

$$H(x, p) \equiv p_\mu \dot{x}^\mu - L(x, \dot{x}) \,, \tag{2.28}$$

which requires us to trade velocities for momenta, $\dot{x} \to p$. We thus move to phase space, which is the space where we can represent all possible states of a physical system. By states of a physical system we refer to the positions and momenta of all objects in the system. These variables are treated as independent variables over the $2n$ dimensional space. We exchange the variables x^μ and $\dot{x}^\mu$ for the phase space variables x^μ and p^μ, where p^μ is the canonical momentum defined in (2.18).

For the relativistic free particle we find employing constraint (2.2),

$$H = m\dot{x}.\dot{x} - m = 0\,, \tag{2.29}$$

and we see that the Hamiltonian vanishes. This is because the action is homogeneous of first degree in velocities,

$$L[\lambda\dot{x}] = \lambda L[\dot{x}]\,. \tag{2.30}$$

Euler's homogeneous function theorem then implies

$$\dot{x}^\mu \frac{\partial L}{\partial \dot{x}^\mu} = \dot{x}\,p = L\ , \tag{2.31}$$

which is (2.29). A vanishing Hamiltonian is characteristic for a parametrization invariant system, i.e. invariance of action under $\tau \to f(\tau), \dot{f} > 0$. This can be shown by a reparametrization of the world line

$$\tau \to \tau'\,, \qquad x^\mu(\tau) \to x^\mu(\tau'(\tau))\,, \tag{2.32}$$

where the mapping $\tau \to \tau'$ is one to one and $d\tau'/d\tau > 0$, which conserves the orientation. The Lagrangian then changes according to

$$L(dx^\mu/d\tau) = L\Big((dx^\mu/d\tau')(d\tau'/d\tau)\Big) = (d\tau'/d\tau)L(dx^\mu/d\tau')\,. \tag{2.33}$$

This is then enough to ensure that the action is unchanged under (2.32) and therefore parametrization invariant as long as the end points remain unchanged.

A solution to this problem of a vanishing Hamiltonian is well known [12, 13]. We use 'gauge fixing' by selecting a time coordinate $T = T(x)$ instead of the proper time τ and as a consequence lose the manifest Lorentz covariance. Each of the time choices has a set of six phase space variables and a Hamiltonian given by a particular component of the canonical momenta p^μ, found by rearranging the dynamical mass-shell constraint $p^2 = m^2$, which is a constant. There are three choices for the time coordinate [12], all of which give equivalent descriptions of the dynamics, but we will concentrate only on the following two choices.

2.4.1 Instant Form

For instant form, we choose Galilei time, $T(x) = x^0 = t$, to parametrise the particle world line. The Lagrangian for the free particle in instant form is

$$L_{\mathrm{IF}} = -m\sqrt{1-v^2} = -\frac{m}{\gamma}\,, \tag{2.34}$$

where we have introduced $\mathbf{v} = d\mathbf{x}/dt$ and $\gamma = 1/\sqrt{1-\mathbf{v}^2}$. The canonical momenta are

$$p_i = \frac{\partial L_{\mathrm{IF}}}{\partial v^i} = \gamma m v^i\ , \tag{2.35}$$

if we compare this to (2.18), we see the reparametrization has introduced a factor of γ.

The instant form Hamiltonian is then given by the Legendre transform

$$H_{\mathrm{IF}} = \mathbf{p}.\mathbf{v} - L_{\mathrm{IF}} = \gamma\, m\, \mathbf{v}^2 + \frac{m}{\gamma} = \gamma\, m(\mathbf{v}^2 + 1/\gamma^2) = \gamma\, m\,. \tag{2.36}$$

We need to trade $\mathbf{v}$ for $\mathbf{p}$, so we rewrite,

$$H_{\mathrm{IF}} = \gamma\, m = \gamma\, m\sqrt{\mathbf{v}^2 + 1/\gamma^2} = \sqrt{\mathbf{p}^2 + m^2} = p^0\ , \tag{2.37}$$

which is hence just a rearrangement of the mass-shell condition (2.19).

Using the example of the free particle we solve the Hamiltonian equations of motion and show that we obtain the same motion found from solving (2.8). The Hamiltonian equations of motion can be defined in terms of

Poisson brackets, a binary operation which also distinguishes a class of coordinate transformations called canonical transformations. In instant form, the Poisson bracket of two phase space functions X and Y is defined by

$$\{X,Y\} = \frac{\partial X}{\partial x^i}\frac{\partial Y}{\partial p_i} - \frac{\partial X}{\partial p_i}\frac{\partial Y}{\partial x^i} \tag{2.38}$$

and the time evolution of a quantity X can then be written in terms of the Poisson bracket as

$$\frac{dX}{dt} = \frac{\partial X}{\partial t} + \{X, H\}\,, \tag{2.39}$$

where we have allowed for an explicit time dependence of X. We move to using the canonical coordinates and find the Hamiltonian equations of motion using (2.37),

$$\frac{dp^i}{dt} = 0\,, \quad \frac{dx^i}{dt} = \frac{p^i}{p^0}\,, \tag{2.40}$$

and the solutions to these equations are,

$$p^i = p_0^i = \text{const}\,, \quad x^i(t) = x_0^i + \frac{p_0^i}{p^0}\,t\,. \tag{2.41}$$

We see that the particle motion is linear in our chosen time coordinate. We can check consistency of this solution with the covariant solution (2.10) by eliminating τ from equation for $x^0 = t$ via

$$\tau(t) = t/u_0^0 \equiv t/\gamma_0\,, \tag{2.42}$$

setting $t_0 = 0$. When this is substituted back into (2.10), we can express the solutions for the remaining directions in terms of the parameter t,

$$x^i(t) = x_0^i + \frac{u_0^i}{u_0^0}\, t\,. \tag{2.43}$$

We can see that (2.41) and (2.43) are in agreement.

2.4.2 Front form

For the front form, we choose the time coordinate $T(x) = x^+ \equiv t + z$. Here the phase space is spanned by the 'longitudinal' coordinate $x^- \equiv t - z$, the 'transverse' coordinates x^i, $i = 1, 2$, collected into the 2-vector $\mathbf{x}^\perp = (x^1, x^2) \equiv (x, y)$, and the conjugate momenta, $p^+ = p^0 + p^3$ and $\mathbf{p}^\perp \equiv (p^1, p^2)$.

The Lagrangian for the free particle in front form is

$$L_{\mathrm{FF}} = -m\sqrt{\frac{dx^-}{dx^+} - \left(\frac{dx^\perp}{dx^+}\right)^2} =: -m\sqrt{w^- - w^\perp w^\perp} =: -m/\eta\,, \tag{2.44}$$

which implies that the canonical momenta are

$$p^+ = -2\,\frac{\partial L_{\mathrm{FF}}}{\partial w^-} = \eta m\ , \quad p^\perp = \frac{\partial L_{\mathrm{FF}}}{\partial w^i} = \eta m w^\perp\ , \tag{2.45}$$

such that

$$w^- = \frac{p_\perp^2 + m^2}{(p^+)^2} \equiv \frac{p^-}{p^+}\ . \tag{2.46}$$

The front form Hamiltonian is then given by the Legendre transformation

$$H_{\mathrm{FF}} = -p^+ w^-/2 + p^\perp w^\perp - L_{\mathrm{FF}} = p^-/2 \, , \tag{2.47}$$

which again can be obtained by rearranging the mass shell constraint (2.19), $p^+ p^- - p^\perp p^\perp = m^2$.

In front form, the Poisson bracket of two phase space functions X and Y is defined by

$$\{X, Y\} = -2 \left(\frac{\partial X}{\partial x^-} \frac{\partial Y}{\partial p_+} - \frac{\partial X}{\partial p_+} \frac{\partial Y}{\partial x^-} \right) + \frac{\partial X}{\partial x^i} \frac{\partial Y}{\partial p_\perp} - \frac{\partial X}{\partial p_\perp} \frac{\partial Y}{\partial x^i} \, , \tag{2.48}$$

and the time evolution of a quantity X can then be written in terms of the Poisson bracket as follows,

$$\frac{dX}{dx^+} = \frac{\partial X}{\partial x^+} + \{X, H\} \, , \tag{2.49}$$

where we have allowed for explicit time dependence of X.

Using the example of the free particle we solve the Hamiltonian equations of motion and show that we obtain the same motion as found from solving (2.8). The Hamiltonian equations for the transverse and longitudinal phase space variables are

$$\begin{aligned} \frac{dp^\perp}{dx^+} &= 0\ , \quad \frac{dx^\perp}{dx^+} = \frac{p^\perp}{p^+}\ , \\ \frac{dp^+}{dx^+} &= 0\ , \quad \frac{dx^-}{dx^+} = \frac{p^-}{p^+}\ . \end{aligned} \tag{2.50}$$

We note that these require $p^+ \neq 0$. The solutions to these equations are

$$\begin{aligned} p^\perp = p_0^\perp = \text{const}\ , &\quad x^\perp(t) = x_0^\perp + \frac{p_0^\perp}{p_0^+}\, x^+ \\ p^+ = p_0^+ = \text{const}\ , &\quad x^-(x^+) = x_0^- + \frac{p_0^-}{p_0^+}\, x^+\ . \end{aligned} \tag{2.51}$$

We see that free particle motion is linear in our chosen time coordinate. We can again check the consistency of the solutions by eliminating τ from the covariant solution (2.10) for x^+, setting $x_0^+ = 0$, which yields

$$\tau(x^+) = x^+/u_0^+\ . \tag{2.52}$$

When this rearrangement is substituted back into (2.10), we can express the solutions for the remaining directions in terms of the parameter x^+,

$$\begin{aligned} x^\perp(x^+) &= x_0^\perp + \frac{u_0^\perp}{u_0^+}\, x^+\ , \\ x^-(x^+) &= x_0^- + \frac{u_0^-}{u_0^+}\, x^+\ . \end{aligned} \tag{2.53}$$

Once again we can see that the solutions (2.51) and (2.53) are in agreement as expected.

2.5 Integrability

A good definition of integrability is given by Bühler, who states that "a mechanical system is called integrable if we can reduce its solution to a sequence of quadratures" [14]. That is, we are able to express the solutions in terms of integrals. In Hamiltonian mechanics, a more formal definition of what is called Liouville integrability can be given as follows. Assuming canonical Poisson brackets,

$$\{p^i, p^j\} = \{x^i, x^j\} = 0\ , \quad \{x^i, p^j\} = \delta_{ij}\ , \quad \forall\, i, j = 1...n\,, \tag{2.54}$$

a dynamical system is Liouville integrable if there exist n independent conserved quantities, Q_i, that are in involution, i.e., their Poisson brackets vanish,

$$\{Q_i, Q_j\} = 0 \quad \forall\, i, j = 1...n\,. \tag{2.55}$$

For integrability we also require that the conserved quantities are functionally independent [15], that is, none of the Q_i's can be written in terms of the others. To test if the conserved quantities are functionally independent we follow [15] and define the set F as an N-vector, where N is the total number of conserved quantities,

$$F \equiv [Q_1(x,p), Q_2(x,p), ...Q_N(x,p)]\,. \tag{2.56}$$

The conserved quantities, Q_i, are functionally independent if the $N \times 2n$ matrix,

$$\left(\frac{\partial F_l}{\partial x^i}, \frac{\partial F_l}{\partial p^i} \right) , \tag{2.57}$$

has rank N in the region where the set of functions Q_i are defined and analytic. If we have $N > 2n$, more conserved quantities then the dimensions of phase space, the set of functions is clearly functionally dependent. In such cases there exist constraints on the conserved quantities which gives us information regarding the independent quantities. In the cases where the energy is conserved and therefore the Hamiltonian does not depend on the chosen time coordinate, the Hamiltonian itself can be one of the quantities, e.g. $Q_1 = H$.

If these conditions are met then the Liouville theorem states that "the equations of motion of a Liouville-integrable system can be solved by quadratures" [16].

2.5.1 Example

As a simple example of integrability we shall consider a system with only one degree of freedom. We write the Hamiltonian of this system as,

$$H(x,p) = \frac{1}{2}p^2 + V(x) \, . \tag{2.58}$$

The Hamiltonian equations for this system are

$$\dot{x} = p, \qquad \dot{p} = -\frac{dV}{dx}\,. \tag{2.59}$$

As the Hamiltonian in time independent, it is a conserved quantity,

$$\frac{1}{2}p^2 + V(x) = E_0\,, \tag{2.60}$$

which we denotes as E_0, a constant energy. We then rearrange to obtain the expression for the momenta,

$$p = \sqrt{2(E_0 - V(x))}\,. \tag{2.61}$$

Using the relationship, $p = dx/dt$, we are able to obtain the solution for the particle trajectory in implicit form,

$$t = \int dx \frac{1}{\sqrt{2(E_0 - V(x))}}\,. \tag{2.62}$$

As long as we are able to evaluate the integral and invert the resulting relationship so that we have $x(t)$, we can recover the explicit solution. These last two steps are not always possible and depend on the function $V(x)$ but we still consider this system integrable.

2.6 Superintegrability

A system that has further k conserved quantities is called superintegrable [17]. If $k = 1$, the system is described as minimally superintegrable and maximally superintegrable if there exist $k = n - 1$ additional conserved quantities. One degree of freedom is left free to allow for the time evolution.

We are interested in finding maximally superintegrable systems because they have many benefits. In particular, where we would usually have a system of differential equations to solve, these are replaced by algebraic expressions, and therefore the particle motion can be found analytically or by algebraic means. There is also a conjecture proposed by Tempesta et al. [18] that for maximally superintegrable classical systems, the associated quantum system is exactly solvable. By quantum solvable we mean that it is possible to find exact solutions to the quantum problem. However, there is currently no proof or counter example that every superintegrable system is necessarily quantum solvable. We also find that for classical systems that are both maximally superintegrable and have finite trajectories, these trajectories are closed and the resulting motion is periodic [19] as superintegrability restricts the trajectories to a subspace of phase space of size $n - k$ for $0 < k < n$.

2.6.1 Example 1: The free relativistic particle

To investigate the integrability of the free relativistic particle we use the instant form of the Hamiltonian. We know from above that there are 10 conserved quantities that are given by the Poincaré generators H, p^i, $L^i = \epsilon^{ijk}M^{jk}$, $K^i = M^{0i}$. The free relativistic particle is an integrable system as any set of three of the momenta are involution. The free particle also has additional Noether charges. However, the additional Noether charges are not all independent and there are constraints on the conserved quantities which reduces the number to five independent conserved quantities. These constraints are, $\mathbf{p}.\mathbf{L} = 0$, $\mathbf{K}.\mathbf{L} = 0$ and $\mathbf{W} := H\mathbf{L} - \mathbf{p} \times \mathbf{K} = 0$, the Pauli-Lubanski vector. So we have shown that altogether the free relativistic particle has 5 independent Q's and hence is a maximally super-integrable system.

2.6.2 Example 2: The Kepler problem

The best known example of superintegrability presumably is the Kepler problem and its quantum equivalent system, the hydrogen atom [20, 21]. The Kepler problem is also one of only two central force potential where all the bounded orbits are closed [22], the other being the harmonic oscillator, which is also superintegrable.

The Hamiltonian for the unit mass ($m = 1$) Kepler problem has the nonrelativistic form

$$H = \frac{\mathbf{p}^2}{2} + V(r)\,, \tag{2.63}$$

with Newton's gravitational potential

$$V(r) = -\frac{\alpha}{r}\,, \quad \alpha = \text{const}\,. \tag{2.64}$$

The equation of motion for the Kepler problem in terms of the position vector $\mathbf{r} = (x, y, z)$ and unit vector $\mathbf{e_r} = \mathbf{r}/r$ is

$$\ddot{\mathbf{r}} = -\frac{\alpha}{r^2}\,\mathbf{e}_r \tag{2.65}$$

and we introduce the triad of unit vectors that correspond to the cylindrical coordinates r, θ, z,

$$\mathbf{e_r} = \begin{pmatrix} \cos\theta \\ \sin\theta \\ 0 \end{pmatrix}, \quad \mathbf{e}_\theta = \begin{pmatrix} -\sin\theta \\ \cos\theta \\ 0 \end{pmatrix}, \quad \mathbf{e_z} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \tag{2.66}$$

The potential, V, for the Kepler problem is spherically symmetric and so we find that the angular momentum $\mathbf{L}$ is conserved, $\mathbf{L} = \text{const}$. This restricts the motion to the xy plane and implies the constant magnitude

$$l := |\mathbf{L}| = r^2\dot{\theta} = \text{const}\,. \tag{2.67}$$

As a consequence we are able to trade the angular speed $\dot{\theta}$ for l which leads to the relationship

$$\mathbf{e_r} = -\frac{r^2}{l}\dot{\mathbf{e}}_\theta \quad (l \neq 0)\,. \tag{2.68}$$

This means that the equation of motion can now be expressed as a total time derivative if we exclude degenerate linear collision orbits ($l = 0$). We therefore get an additional three conservation laws,

$$\mathbf{v} - \frac{\alpha}{l}\mathbf{e}_\theta \equiv \mathbf{V}_0 = \text{const}\,. \tag{2.69}$$

The conserved quantity, $\mathbf{V}_0$, a velocity, is sometimes referred to as the Hamiltonian vector [23]. The result (2.69) can now be rearranged to give the Kepler *hodograph*, the velocity locus $\mathbf{v} = \mathbf{v}(t)$, or the first integral of the equations of motion,

$$\mathbf{v}(t) = \mathbf{V}_0 + \frac{\alpha}{l}\,\mathbf{e}_\theta(t) \equiv \mathbf{V}_0 + V\,\mathbf{e}_\theta(t)\;. \tag{2.70}$$

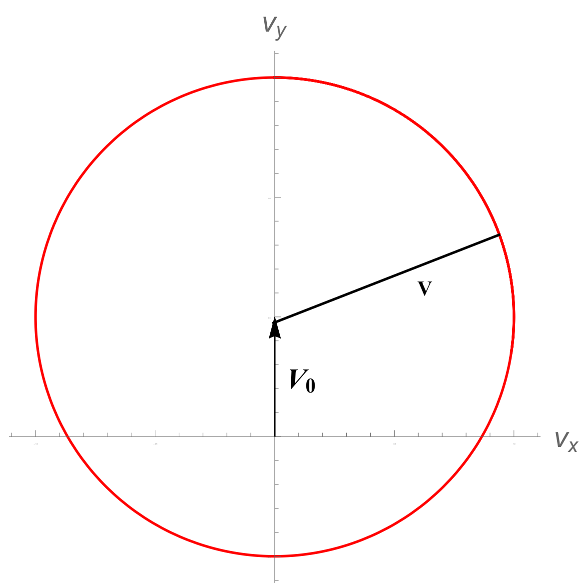


Figure 2.3: Circular Kepler hodograph with center $\mathbf{V}_0$ and radius V.

The central force is a potential force, hence the total energy of the system is conserved and can be represented through $\mathbf{V}_0$ and $V = \alpha/l$ according to

$$E = \frac{v^2}{2} - \frac{\alpha}{r} = \frac{1}{2}(V_0^2 - V^2) = \text{const} \tag{2.71}$$

In total, this gives us seven conserved quantities but (2.71) expresses the energy in terms of $\mathbf{V}_0$ and l. Furthermore, we see that from (2.70) that $\mathbf{V}_0$ is in the xy plane whilst $\mathbf{L}$ is along z, which implies another constraint,

$$\mathbf{V}_0 \cdot \mathbf{L} = 0\,. \tag{2.72}$$

As a result there are only five functionally independent conserved quantities of the system and therefore it is maximally superintegrable. The conserved quantities of the Kepler problem are normally discussed in terms of the Runge-Lenz vector, $\mathbf{K}$, which is defined as

$$\mathbf{K} \equiv \mathbf{V}_0 \times \mathbf{L} = \text{const} \tag{2.73}$$

and is located in the plane perpendicular to Hamilton's vector $\mathbf{V}_0$, see Fig. 2.4. The conservation of the Runge-Lenz vector means that there is no precession of the orbit.

As the Kepler problem is maximally superintegrable, the orbits can be found algebraically. To do so we follow the discussion of Milnor [24] and chose the coordinate system such that $\mathbf{V}_0 = V_0\mathbf{e_y}$, so the hodograph (2.70)

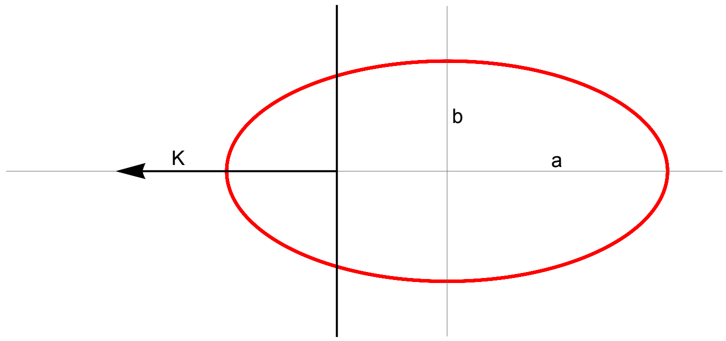


Figure 2.4: Direction of the Runge-Lenz vector, $\mathbf{K}$, relative to the Kepler ellipse, where a is the semi-major axis and b is the semi-minor axis.

becomes

$$\mathbf{v} = V_0 \mathbf{e_y} + V \mathbf{e}_\theta \,. \tag{2.74}$$

The angular momentum, $\mathbf{L} = \mathbf{r} \times \mathbf{v} = l\mathbf{e}_z$, thus has magnitude

$$l = r(V + V_0 \cos\theta) \,. \tag{2.75}$$

The parametric representation of the Kepler orbit is then found by rearranging this expression using $V = \alpha/l$,

$$r = \frac{l^2/\alpha}{1 + (V_0/V)\cos\theta} \,. \tag{2.76}$$

This identifies the Kepler orbit as a conic section with eccentricity $\varepsilon := V_0/V$. The value of this parameter controls the shape of the orbit, either a circle, ellipse, parabola or a hyperbola for $\varepsilon = 0, 0 < \varepsilon < 1, \varepsilon = 1,$ or $\varepsilon > 1$, respectively. As anticipated, we have obtained the orbit for the Kepler ellipse without performing any integration. The solutions to the equation of motion were found using purely algebraic means through the use of the conserved quantities.

We will now present further examples of integrable and superintegrable systems for a relativistic particle interacting with different background fields. We shall investigate both vector and scalar background fields.

Chapter 3

Vector Fields

In this chapter we will discuss the action and equations of motion for a relativistic particle in a vector background field. We will investigate the conserved quantities of different vector fields and classify the integrability of the fields.

3.1 Introduction

A vector field is a field that at every point has a direction and magnitude. The field is defined by a set of n ordered scalar functions. We can visualise the field as a collection of arrows that are attached to a point in space-time with a given magnitude and direction. We will define our 4-vector field as a gauge potential $A^\mu = (A^0, A^1, A^2, A^3) = (A^0, \mathbf{A})$. Under a Lorentz transformation, a 4-vector transforms as,

$$A^{\mu}(x) \to \Lambda^{\mu}{}_{\nu} A^{\nu}(\Lambda^{-1}x)\,. \tag{3.1}$$

3.1.1 Gauge transformations and gauge invariance

In electrodynamics, we can express the electric field and the magnetic field in terms of field strength tensor $F^{\mu\nu} = \partial^{\mu}A^{\nu} - \partial^{\nu}A^{\mu}$. These fields form our gauge potential A^{μ}. The term gauge invariance refers to the property that we can describe the same electromagnetic field $F^{\mu\nu}$ by many different gauge potentials but the observables of the electric and magnetic fields remain the same. This itself is a symmetry which allows us the freedom to choose a coordinate system to work in. Therefore the dynamics of a charged particle in an electromagnetic background field remain the same and do not depend on our choice of gauge potential A^{μ}. This allows us to work with a more convenient choice of A^{μ}.

To change from one gauge potential A^{μ} to another A'^{μ}, we apply a gauge transformation,

$$A'^{\mu} = A^{\mu} + \partial^{\mu}\lambda\,. \tag{3.2}$$

We now calculate $F'^{\mu\nu}$,

$$\partial^{\mu}A^{\nu} + \partial^{\mu}\partial^{\nu}\lambda - \partial^{\nu}A^{\mu} - \partial^{\mu}\partial^{\nu}\lambda = F^{\mu\nu} + \partial^{\mu}\partial^{\nu}\lambda - \partial^{\mu}\partial^{\nu}\lambda\,. \tag{3.3}$$

As the last two terms cancel we see that the observable electromagnetic field

$F^{\mu\nu}$ remains unchanged. Hence the electric and magnetic fields are gauge invariant.

3.1.2 Dynamics in a vector field

The dynamics of a relativistic particle of charge e and mass m, moving in a background field A^μ are described by the action,

$$S = -\int d\tau (m + eA(x) \cdot \dot{x}) \,. \tag{3.4}$$

We vary the action according to the principle of least action and we recover the Lorentz force equation of motion,

$$m\ddot{x}^\mu = e(\partial^\mu A^\nu - \partial^\nu A^\mu)\dot{x}_\nu = eF^{\mu\nu}\dot{x}_\nu \,, \tag{3.5}$$

where $F^{\mu\nu}$ is the electromagnetic field strength tensor which is gauge invariant. We can choose to give up this gauge invariance to rewrite this as

$$\dot{P}^\mu = e\dot{x}_\nu \partial^\mu A^\nu \,, \tag{3.6}$$

where P^μ is the canonical momentum $P^\mu = m\dot{x}^\mu + eA^\mu$.

For a free particle, the Poincaré symmetry is generated by the 10 'Noether charges', $\xi.P = \xi_\mu P^\mu$. These gain a proper time dependence when in the presence of the background field A^μ. This is found using the equation of motion (3.6),

$$\frac{d}{d\tau}\xi^\mu P_\mu = \frac{m}{2}\dot{x}^\mu(\partial_\mu \xi^\nu + \partial_\nu \xi^\mu)\dot{x}_\nu + e\dot{x}^\mu \mathcal{L}_\xi A_\mu\,. \tag{3.7}$$

The first term vanishes due to the Killing equation, reducing to

$$\frac{d}{d\tau}\xi^\mu P_\mu = e\dot{x}^\mu \mathcal{L}_\xi A_\mu\,. \tag{3.8}$$

where $\mathcal{L}_\xi$ is the Lie derivative of the background field,

$$\mathcal{L}_\xi A_\mu \equiv \xi^\nu \partial_\nu A_\mu + A_\nu \partial_\mu \xi^\nu\,. \tag{3.9}$$

If the background field A^μ is invariant under the action of the Lie derivative up to a gauge transformation,Λ,

$$\mathcal{L}_\xi A_\mu = \partial_\mu \Lambda\,, \tag{3.10}$$

we say that it is a symmetric gauge field [25]. For symmetric gauge fields, (3.8) is a total derivative which can be integrated directly to give the conserved quantity

$$Q_\xi = \xi \cdot P - e\Lambda\,. \tag{3.11}$$

It follows that if the background field possesses a Poincaré symmetry then there exists a corresponding conserved quantity in the particle motion. If we can find a background field that has enough Poincaré symmetries then the particle motion will be (super) integrable. In addition to Poincaré symmetry, the background field may also possess hidden symmetries or symmetries that are of a higher order in the momenta P^μ.

3.2 Simple examples

The first examples we shall consider are those where the background field are constant in space and time. The trajectories for constant fields fall into four distinct cases as shown in [26], and account for all possible forms of constant field. Here we will investigate the integrability of each case.

3.2.1 Elliptic Motion

Elliptic motion arises from a particle moving in a constant magnetic background field. To model this we can choose the gauge

$$A^\mu = (0, By, 0, 0)\,, \tag{3.12}$$

where B is the constant magnetic field strength. The field strength tensor has only two non-zero components, $F^{21} = -F^{12}$, where $F^{12} = B$. The Hamiltonian for this system is then defined as

$$H = \sqrt{m^2 + P_2^2 + P_3^2 + (P^1 - eBy)^2}\,. \tag{3.13}$$

Dynamics in a magnetic field is an example of an autonomous system and therefore the Hamiltonian is a conserved quantity. In addition to this the two spatial momenta P^1 and P^3 are also conserved. These three conserved quantities are in involution and therefore we conclude that the constant magnetic field is integrable. We now show that this field also has an additional two Poincaré symmetries, the boost M^{03} and the rotation about the z axis, M^{12}.

The boost M^{03} is trivially conserved due to the conservation of the Hamiltonian, P^0, and the momentum in the P^3 direction,

$$\dot{M}^{03} = \dot{x}^0 P^3 + x^0 \dot{P}^3 - \dot{z} P^0 - z \dot{P}^3 = \frac{p^0}{m} p^3 - \frac{p^3}{m} p^0 = 0. \tag{3.14}$$

The rotation about the z axis, however requires a modification, which we can identify from taking the derivative with respect to the proper time τ,

$$\dot{M}^{12} = \dot{x} P^2 + x \dot{P}^2 - \dot{y} P^1 - y \dot{P}^1 = \frac{eB}{2m} \frac{d}{d\tau} (x^2 - y^2) \,. \tag{3.15}$$

We rearrange this to give an expression that is equal to zero. The conserved quantity, a modified rotation, is then given by integrating,

$$L_z = M^{12} - \frac{eB}{2m} (x^2 - y^2), \tag{3.16}$$

These five conserved quantities, $\{H, P^1, P^3, M^{03}, L_z\}$, are independent and we conclude that the constant magnetic field is maximally superintegrable.

The solutions to the equations of motion are now as follows. The dynamics in the trivial plane can be found from the conserved quantity M^{03},

$$z(t) = \frac{tP^3 - M^{03}}{P^0}\,, \tag{3.17}$$

and find that the dynamics for z are linear in t. For the dynamics in the non-trivial plane, we use the relationship between the canonical and mechanical momenta, $P^\mu = p^\mu + eA^\mu$, therefore we have,

$$\begin{aligned} \dot{p}^x =& -\frac{eB}{m} p_y\,, \\ \dot{p}^y =& \frac{eB}{m} p_x\,. \end{aligned} \tag{3.18}$$

This coupled system describes a two dimensional harmonic oscillator, to which the solution is,

$$\begin{aligned} p_x =& p_{x0}\cos(\Omega_B\tau) - p_{y0}\sin(\Omega_B\tau)\,, \\ p_y =& p_{y0}\cos(\Omega_B\tau) + p_{x0}\sin(\Omega_B\tau)\,, \end{aligned} \tag{3.19}$$

where we have defined the Larmor frequency $\Omega_B = eB/m$. The particle trajectory are then given by one further integration,

$$\begin{aligned} x - x_0 =& \frac{p_{x0}}{m\Omega_B}\sin(\Omega_B\tau) + \frac{p_{y0}}{m\Omega_B}\left(\cos(\Omega_B\tau) - 1\right)\,, \\ y - y_0 =& \frac{p_{y0}}{m\Omega_B}\sin(\Omega_B\tau) - \frac{p_{x0}}{m\Omega_B}\left(\cos(\Omega_B\tau) - 1\right)\,. \end{aligned} \tag{3.20}$$

The trajectory of a particle in a constant magnetic background field is shown below 3.1.

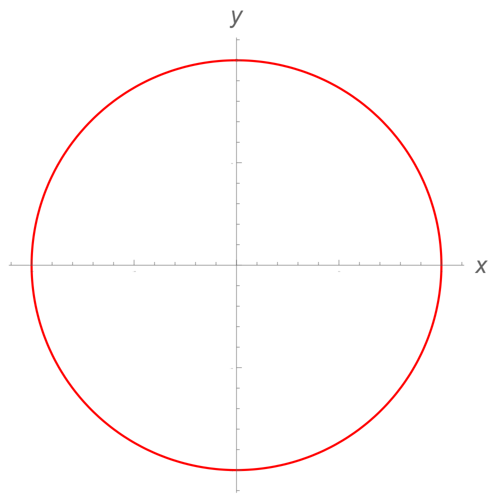


Figure 3.1: Particle trajectory of a particle in a constant magnetic field where the particle is initially at rest, with $P^3 = 0$ and $B = m = 1$.

Figure 3.1 shows the trajectory of a particle in a constant magnetic field. In the xy plane the path of the particle follows a circle with radius $R = \sqrt{p_{x0}^2 + p_{y0}^2}/m\Omega_B$ which is centered at,

$$\vec{x}_\perp = \left(x_0 - \frac{p_{y0}}{m\Omega_B}, y_0 + \frac{p_{x0}}{m\Omega_B} \right) . \tag{3.21}$$

3.2.2 Hyperbolic Motion

Hyperbolic motion arises from a particle moving in a constant electric background field. To model this we can choose the gauge,

$$A^\mu = (0, 0, 0, -Et) , \tag{3.22}$$

where E is the constant electric field strength. The associated field strength tensor has only the two non-zero components, $F^{03} = -F^{30}$, where $F^{30} = E$. The Hamiltonian for this system is then defined as,

$$H = \sqrt{m^2 + P_\perp^2 + (P^3 + eEt)^2}\,. \tag{3.23}$$

The electric field is an example where the Hamiltonian depends explicitly on the chosen time coordinate. Due to this, all three spatial momenta are conserved. The three conserved components are in involution and therefore the constant electric field is integrable. We now show that the field has an additional two Poincaré symmetries, the boost M^{03} and the rotation about the z axis, M^{12}.

The rotation about the z axis, M^{12}, is trivially conserved due to the conservation of the momentum in the P^1 and P^2 directions,

$$\dot{M}^{12} = \dot{x}P^2 + x\dot{P}^2 - \dot{y}P^1 - y\dot{P}^1 = \frac{p^1}{m}p^2 - \frac{p^2}{m}p^1 = 0\,. \tag{3.24}$$

The boost M^{03} is not trivially conserved and requires a modification which we once again identify by taking the derivative,

$$\dot{M}^{03} = \dot{t}P^3 + t\dot{P}^3 - \dot{z}P^0 - z\dot{P}^0 = -\frac{eE}{2}\frac{d}{d\tau}(t^2 + z^2). \tag{3.25}$$

We rearrange this to give an expression that is equal to zero. The conserved quantity is then given by integration this expression,

$$T^{03} = M^{03} + \frac{eE}{2}(t^2 + z^2)\,. \tag{3.26}$$

The five conserved quantities, $\{P^1, P^2, P^3, M^{12}, T^{03}\}$, are independent and we conclude that the constant electric field is maximally superintegrable. The particle trajectory is then found as follows. From the conserved momenta in the transverse directions, we have linear dynamics in this plane,

$$\vec{x}_\perp = \vec{x}_{\perp 0} + \frac{\vec{p}_{\perp 0}}{m}\tau\,. \tag{3.27}$$

To solve for the dynamics in the non-trivial plane, we one again use the mechanical momenta and the conservation of the canonical momentum in the z direction we find,

$$\ddot{p}_0 = \Omega_E \dot{p}_z = \Omega_E^2 p_0\,, \tag{3.28}$$

where $\Omega_E = eE/m$. The solutions are then,

$$\begin{aligned} p_0 =& p_0^0 \cosh(\Omega_E \tau) + p_{z0} \sinh(\Omega_E \tau)\,, \\ p_z =& p_{z0} \cosh(\Omega_E \tau) + p_0^0 \sinh(\Omega_E \tau)\,. \end{aligned} \tag{3.29}$$

A further integration gives the particle trajectories,

$$\begin{aligned} t =& \frac{P p_0^0}{m\Omega_E} \sinh(\Omega_E \tau) + \frac{p_{z0}}{m\Omega_E} \cosh(\Omega_E \tau)\,, \\ z =& \frac{p_{z0}}{m\Omega_E} \sinh(\Omega_E \tau) + \frac{p_0^0}{m\Omega_E} \cosh(\Omega_E \tau)\,. \end{aligned} \tag{3.30}$$

The trajectory of a particle in a constant electric background field is shown below 3.2.

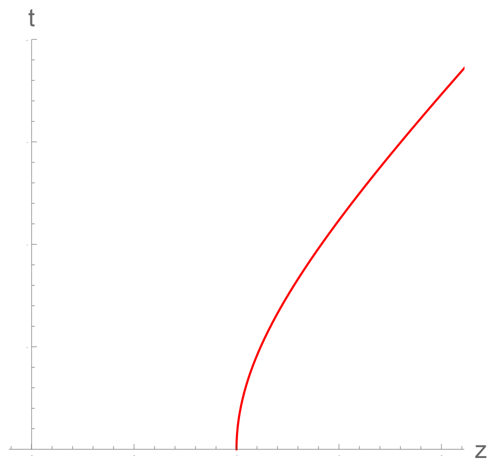


Figure 3.2: Particle trajectory of a particle in a constant electric field, particle initially at rest, at $z = z_0$, no dynamics in the xy plane, $E = m = 1$.

We see from the figure that the field accelerates the particle until it asymptotically approaches the speed of light.

3.2.3 Loxodromic Motion

Loxodromic motion arises from a particle moving in a background field created from a linear combination of the elliptic and hyperbolic cases. To model this we may choose the gauge,

$$A^\mu = (0, By, 0, -Et)\,. \tag{3.31}$$

For this case the gauge field is no longer univariate. The field is formed by a magnetic field and electric field that are parallel to each other. The field strength tensor now has four non-zero components, $F^{21} = -F^{12}$, which encodes the magnetic field and $F^{03} = -F^{30}$ which encodes the electric field. The Hamiltonian for this field is,

$$H = \sqrt{m^2 + (P^1 - eBy)^2 + P^{2^2} + (P^3 + eEt)^2}\,. \tag{3.32}$$

As a combination of the elliptic and hyperbolic cases, it inherits the two conserved momenta, P^1 and P^3, that both the individual cases had in common. The loss of one conserved momenta is connected to the gauge field no longer being univariate. The parallel field also inherits the two modified Poincaré symmetries,

$$\begin{aligned} L_z &= M^{12} - \frac{eB}{2}(x^2 - y^2)\,, \\ T^{03} &= M^{03} + \frac{eE}{2}(t^2 + z^2)\,. \end{aligned} \tag{3.33}$$

For a system to be integrable, we require there to be three independent conserved quantities that are also in involution. Although this case has four conserved quantities, no three of them are in involution and therefore we cannot classify the system as integrable. Yet the system may still be classed as minimally superintegrable as there is no know proof that every superintegrable system is necessarily integrable. However, we are still able to resolve the particle trajectory. The dynamics for the parallel field separate into the two planes, tz and xy. The trajectories in each of the planes are given by those found in the individual field cases. In the tz plane we have hyperbolic motion, (3.30), and in the xy plane we have elliptic motion, (3.20). The trajectory of a particle in a constant parallel field background field is shown below 3.3.

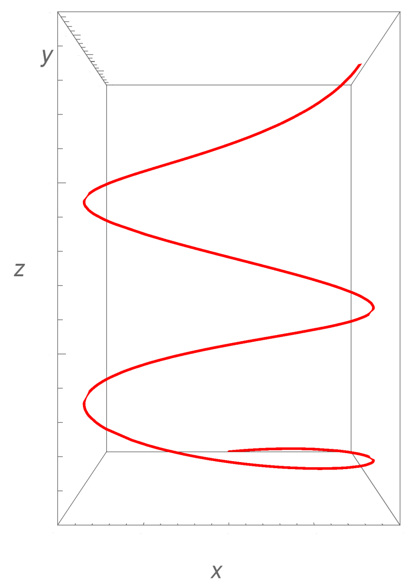


Figure 3.3: Particle trajectory of a particle in a constant parallel field, particle initially at rest, at the origin, $E = B = m = 1$.

From the figure we see the circular motion inherited from the magnetic field in the xy plane and the hyperbolic motion inherited from the electric field in the z direction.

3.2.4 Parabolic Motion

Parabolic motion arises from a particle moving in a constant crossed field. We create this field from a constant magnetic field and a constant electric field, of equal strength, that are orthogonal to each other. We can still express this field using a univariate gauge potential by moving to light-front co-ordinates and re-ordering the components so that it has the form,

$$A^\mu = (A^+, A^1, A^2, A^-) = (0, 0, Fx^+, 0) \tag{3.34}$$

where F is the field strength and the associated field strength tensor has two non-zero components, $F^{+2} = -F^{2+} = F$. The Hamiltonian for this is defined as

$$H = \frac{m^2 + P_1^2 + (P_2 - e\,Fx^+)^2}{P^+}\,. \tag{3.35}$$

As with the constant electric field, the Hamiltonian depends explicitly on our chosen time parameter x^+. This leads to the conservation of the momentum in the P^1, P^2 and the P^+ directions. The three conserved components are in involution and thus the constant crossed field is integrable. We shall now demonstrate that the crossed field has a further two Poincaré symmetries, the two null rotations, M^{+i}. The null rotation, M^{+1}, is trivially conserved due to the conservation of the P^1 momenta,

$$\dot{M}^{+1} = \dot{x}^+ P^1 + x^+ \dot{P}^1 - \dot{x} P^+ - x\dot{P}^+ = \frac{p^+}{m} p^1 - \frac{p^1}{m} p^+ = 0\,. \tag{3.36}$$

The second null rotation, M^{+1}, is not trivially conserved and requires a modification. We again identify this from taking the derivative with respect to τ,

$$\dot{M}^{+2} = \dot{x}^{+}P^{2} + x^{+}\dot{P}^{2} - \dot{y}P^{+} - y\dot{P}^{+} = -\frac{d}{d\tau}\left(\frac{P^{+}\,eF}{2m}x^{+^{2}}\right). \tag{3.37}$$

We rewrite the gauge term as the integral of the derivative of the gauge term, so that the conserved quantity is

$$T^{+2} = M^{+2} + \frac{F\,e\,P^{+}}{2m}x^{+^{2}}. \tag{3.38}$$

The five conserved quantities, $\{P^{+}, P^{1}, P^{2}, M^{+1}, T^{+2}\}$, are independent and we conclude that the constant crossed field is maximally superintegrable. The solution to the Hamiltonian equations of motion are as follows. The transverse trajectories can be obtained algebraically from the two null rotations,

$$\begin{aligned} x(x^{+}) &= \frac{P^{1}x^{+} - M^{+1}}{P^{+}} \\ y(x^{+}) &= \frac{1}{P^{+}}\left(P^{2}x^{+} + \frac{FeP^{+}}{2m}x^{+^{2}} - T^{+2}\right). \end{aligned} \tag{3.39}$$

In the x direction we have linear dynamics. In the y direction we see the parabolic motion. The final direction, x^{-}, is given by

$$x^{-}(x^{+}) = \frac{1}{P^{+}}(m^{2}x^{+} + P_{1}^{2}x^{+} + P_{2}^{2}x^{+} - F\,e\,P_{2}x^{+^{2}} + \frac{F^{2}\,e^{2}\,x^{+^{3}}}{3}) \tag{3.40}$$

The trajectory of a particle in a constant crossed field is shown below 3.4.

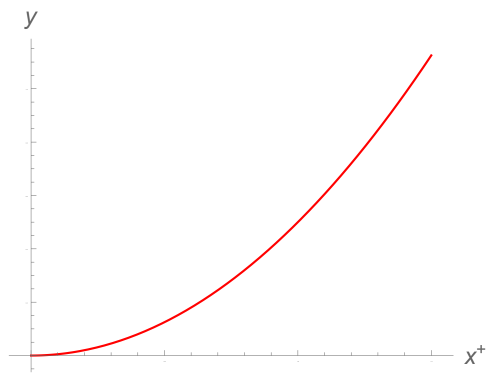


Figure 3.4: Transvere particle trajectory $y(x^+)$ of a particle in a constant crossed field, particle initially at rest, at the origin, $F = m = 1$.

3.3 Further examples

The previous examples are all instances where our choice of gauge has led to creating an $F^{\mu\nu}$ with no spacetime dependence, a constant field. The next simplest case is to consider background fields that depend on a single spacetime coordinate. Therefore, in general, the field has the form $F^{\mu\nu} = F^{\mu\nu}(l.x)$, where the vector l is one of the three choices, space-like ($l^2 > 0$), time-like ($l^2 < 0$) or light-like ($l^2 = 0$).

3.3.1 Plane Wave

The usual model of a laser in high intensity laser matter interactions is a plane wave. This corresponds to the choice of l to be a light-like vector where $l.x$ defines the wavephase φ. It is common in field theory applications for the plane wave to depend on our choice of time, therefore, we choose the gauge,

$$A^\mu = (0, f_1'(x^+), f_2'(x^+), 0)\,. \tag{3.41}$$

Due to the translational invariance of (3.41), we find that the following three canonical momenta are conserved, P^1, P^2 and P^+. These three components of momenta are in involution and thus the plane wave is integrable [8]. In addition to these, we can show that the plane wave has a further two Poincaré symmetries, the two null rotations M^{+i} [27]. These two null rotations are not trivially conserved, therefore they include a gauge term,

$$T^{+i} = M^{+i} - \frac{eP^i}{m} f_i(x^+)\,. \tag{3.42}$$

The five conserved quantities, $\{P^1, P^2, P^+, T^{+1}, T^{+2}\}$ are independent and and we therefore conclude that the plane wave is maximally superintegrable. From here we are able to solve the Hamiltonian equations. The transverse trajectories are found algebraically from the two conserved null rotations,

$$\begin{aligned} x(x^+) &= \frac{x^+P^1 - (eP^1/m)f_1(x^+) - T^{+1}}{P^+} \\ y(x^+) &= \frac{x^+P^2 - (eP^2/m)f_2(x^+) - T^{+2}}{P^+}\,. \end{aligned} \tag{3.43}$$

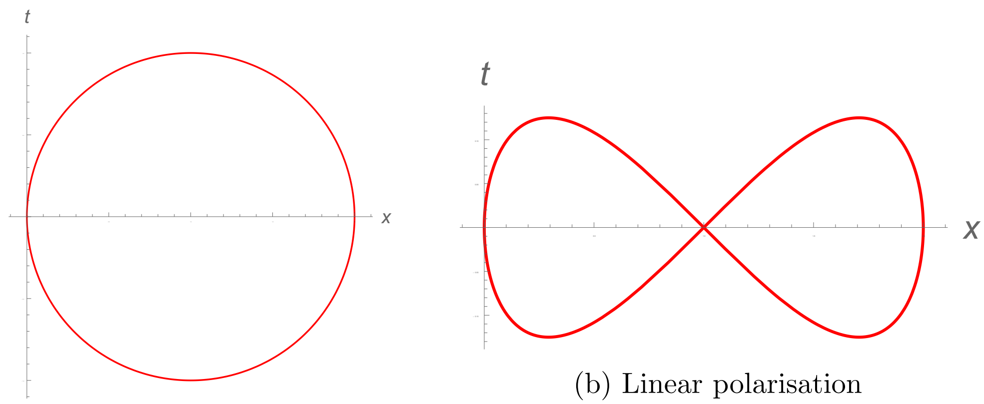
The trajectory for the remaining direction, x^-, is given by solving the Hamiltonian equation,

$$\frac{dx^-}{dx^+} = \frac{m^2 + (P^1 - ef_1'(x^+))^2 + (P^2 - ef_2'(x^+))^2}{{P^+}^2}\,. \tag{3.44}$$

Maximally superintegrable systems have closed orbits [19]. For the plane wave, we see these closed orbits in the average rest frame of the particle. The average rest frame is defined as the frame in which the average momentum of the particle is zero. For a plane wave, this implies that the particle does not experience any longitudinal drift in the direction of the laser propagation. We find this frame by firstly determining the average of the momentum which is given by

$$q^\mu = p_0^\mu - \frac{e^2\langle A^2\rangle k^2}{2l.p_0}\,, \tag{3.45}$$

where the angled brackets denote the time average with respect to the mechanical momenta. We now require this quasi-3-momentum to vanish, $\vec{q} = 0$. Figure 3.5 below shows the particle motion in the average rest frame for both linear and circular polarisation.



(a) Circular polarisation

(b) Linear polarisation

Figure 3.5: Particle trajectory in the average rest frame for a particle moving in a plane wave

3.3.2 Plane Wave plus Constant Electric Field

We now consider an extension of the plane wave by adding a constant longitudinal electric field, ϵE, where ϵ is the amplitude of the longitudinal electric field and typically $\epsilon \ll 1$. The parameter E is the laser field amplitude. This can be used as a simplistic model for a laser propagating through a plasma [28] which creates a space charge ϵE like seen in a capacitor. To model this, we choose the gauge field with the same ordering as (3.34),

$$A^{\mu} = (\epsilon E x^{+}, f_1'(x^{+}), f_2'(x^{+}), 0)\,. \tag{3.46}$$

The Hamiltonian is then given by,

$$H = \frac{m^2 + (P^1 - f_1'(x^{+}))^2 + (P^2 - f_2'(x^{+}))^2}{P^{+} - \epsilon E x^{+}}\,. \tag{3.47}$$

This field retains the translational invariance of the plane wave. This leads to the conservation of the canonical momenta, P^1, P^2 and P^+. As with the plane wave, this system is integrable as these components of momentum are in involution. The introduction of the longitudinal constant electric field results in the loss of the two null rotations.

As the system is still integrable, the solution to the equations of motion proceeds as follows. As the three components of momenta are conserved, these can be integrated directly to give

$$P^+ = p_0^+ - e(A^+ - A_0^+), \qquad P^i = p_0^i - e(A^i - A_0^i)\,. \tag{3.48}$$

We can adopt $A_0^+ = 0$ without any loss of generality, so that the momenta in the P^+ direction is,

$$p^+ = p_0^+ - e\epsilon E x^+\,. \tag{3.49}$$

We see that in contrast to the plane wave, the mechanical momentum has a linear dependence on the choice of time coordinate, x^+. We can investigate this further by taking the derivative with respect to the proper time,

$$\dot{p}^+ = -e\,\epsilon E\dot{x}^+ = \frac{-e\,\epsilon E}{m}p^+\,. \tag{3.50}$$

We find that hyperbolic motion is present. This is to be expected, due to the addition of the constant electric field.

The most convenient method of resolving for the transverse directions is by way of the Hamiltonian equations,

$$\frac{dx^\perp}{dx^+} = \frac{P^\perp}{P^+} = \frac{p_0^\perp}{p_0^+ - e\,\epsilon E x^+} - \frac{A^\perp(x^+)}{p_0^+ - e\,\epsilon E x^+}\,. \tag{3.51}$$

We choose f_1 and f_2 to be trigonometric functions, $f_1' = F\sin(\omega x^+)/\omega$ and $f_2' = F\cos(\omega x^+)/\omega$, the solutions to (3.51) are then given in terms of exponential sine and cosine functions [29],

$$\begin{aligned} x(x^+) &= x(0) + \frac{1}{e\,\epsilon E\,\omega}\left(-p_0^1\,\omega\ln(\omega[p_0^+ - e\,\epsilon E x^+]) + e\,F\,\mathrm{S}_1\left(\omega x^+, \vartheta p_0^+\right)\right), \\ y(x^+) &= y(0) + \frac{1}{e\,\epsilon E\,\omega}\left(-p_0^2\,\omega\ln(\omega[p_0^+ - e\,\epsilon E x^+]) + e\,F\,\mathrm{C}_1\left(\omega x^+, \vartheta p_0^+\right)\right). \end{aligned} \tag{3.52}$$

We define the functions S_1 and C_1 as,

$$\mathrm{S}_1(\omega x^+, \vartheta p_0^+) \equiv \cos(\vartheta p_0^+)\,\mathrm{Si}(u) + \sin(\vartheta p_0^+)\,\mathrm{Ci}(u)\,, \tag{3.53}$$

and

$$\mathrm{C}_1(\omega x^+, \vartheta p_0^+) \equiv \cos(\vartheta p_0^+)\,\mathrm{Ci}(u) - \sin(\vartheta p_0^+)\,\mathrm{Si}(u)\,, \tag{3.54}$$

where we have defined the quantities $\vartheta = \omega/e\,\epsilon E$ and $u = \omega(x^+ - p_0^+/e\,\epsilon E)$ for ease of notation. The remaining direction is found by solving the Hamiltonian equation,

$$\frac{dx^-}{dx^+} = -\frac{H}{P^+}\,. \tag{3.55}$$

To integrate this expression, we make used of equation 2.641.2 in [30] and find the trajectory to be,

$$x^-(x^+) = x_0^- + \frac{m^2 + p_{\perp 0}^2 + E^2F^2}{e\,\epsilon E(p_0^+ - e\,\epsilon E x^+)} - \frac{2eF}{e\,\epsilon E}\left[\vartheta\,\mathrm{C}_1 - \frac{\sin(\omega x^+)}{p_0^+ - e\,\epsilon E x^+} - \left(\frac{\cos(\omega x^+)}{p_0^+ - e\,\epsilon E x^+} + \vartheta\,\mathrm{S}_1\right)\right] \tag{3.56}$$

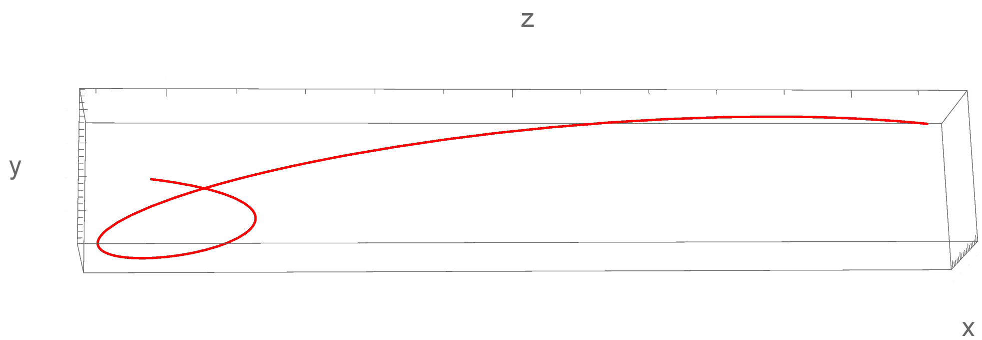


Figure 3.6: Trajectory of a charged particle in a plane wave plus constant electric field, $p_0^1 = p_0^2 = 0$, $m = e = 1$, $\omega = 1.5$ and $\epsilon = 0.00001$

We see from figure 3.6 that the particle has plane wave orbits at short times and hyperbolic motion at large times. Varying the ratio ϵ of the longitudinal to laser electric field causes the onset of the hyperbolic motion to move to earlier or later times.

3.3.3 Undulator

The previous two examples are both examples where our choice of the vector l has been light-like. We now present an example where the choice of l is space-like, the undulator.

Planar Undulator

We shall first consider the planar or linear undulator which can be described by a gauge with only a single entry,

$$A^\mu = (0, \frac{B_0}{k}\sin(kz), 0, 0)\,, \tag{3.57}$$

In instant form, the Hamiltonian is then given by,

$$H = \sqrt{m^2 + (P^1 - (B_0/k)\sin(kz))^2 + P_2^2 + P_3^2}\,. \tag{3.58}$$

As with the constant magnetic field, the undulator is an example of an autonomous system and therefore the Hamiltonian is a conserved quantity. In addition to the Hamiltonian, the two spatial momenta P^1 and P^2 are conserved. The Hamiltonian and the two components of momenta are in involution and therefore the undulator is integrable. The planar undulator has an additional symmetry, M^{12}, which makes the system minimally superintegrable. The rotation about z is conserved but it requires a modification,

$$\dot{M}^{12} = \dot{x}P^2 + x\dot{P}^2 - \dot{y}P^1 - y\dot{P}^1 = \frac{d}{d\tau}\int dz\, \frac{e\, p^2 B_0}{mk}\sin(kz)\,. \tag{3.59}$$

We rewrite the gauge term as the integral of the derivative of the gauge term, so that the conserved quantity is

$$L_z = M^{12} - \int dz\, \frac{e\, p^2 B_0}{mk}\sin(kz)\,. \tag{3.60}$$

To solve the equations of motion, we use the three conserved canonical momenta, P^0, P^1 and P^2 and setting $P^1 = 0$ without any loss of generality, we can immediately integrate these for the mechanical momenta,

$$\begin{aligned}
p^0 &= \gamma_0 m \\
p^1 &= -(eB_0/k)\sin(kz) \\
p^2 &= p_0^2
\end{aligned} \tag{3.61}$$

To find the remaining component of momenta, we use the mass-shell relation and rearrange to find,

$$p_z = [\gamma_0^2 m^2 2 - e^2 A_1^2 - m^2]^{\frac{1}{2}}. \tag{3.62}$$

When evaluating the momentum here, the initial value of this component is found to be

$$p_{z,0} = [\gamma_0^2 m^2 c^2 - m^2 c^2]^{\frac{1}{2}} = \gamma_0 mc\beta\,, \tag{3.63}$$

so we find the z component of momenta to be

$$p^3 = p_0^3\left[1 - K^2\sin^2(kz)\right] \tag{3.64}$$

where K is the undulator parameter $K = eB_0/km$. The energy is conserved in the undulator and we are able to trade the proper time τ for the parameter t through the relationship,

$$\frac{p_z}{m} = \frac{dz}{d\tau} = \frac{\gamma_0 dz}{dt}. \tag{3.65}$$

. This causes the equation of the z trajectory to become a separable ordinary differential equation, with solution

$$z(t) = \frac{\mathrm{am}(B_0 k t, L)}{k} \tag{3.66}$$

where am is the amplitude of an elliptic function and $L = K/\gamma_0 B_0$. The gauge field can now be expressed in terms of the new time parameter t,

$$A^1 = k\, \mathrm{sn}(B_0 k t, L)\,, \tag{3.67}$$

where sn is the Jacobi sine amplitude. The mechanical momentum for x is now

$$p^1 = -\gamma_0 B_0 m L\, \mathrm{sn}(B_0 k t, L) \tag{3.68}$$

which, when performing one further integration, gives the x trajectory as

$$x = x_0 + \gamma_0 B_0 \left(\ln(1 - L) - \ln[\mathrm{dn}(B_0 k t, L) - L\, \mathrm{cn}(B_0 k t, L)] \right) \tag{3.69}$$

where cn is the Jacobi cosine amplitude and dn is the delta amplitude.

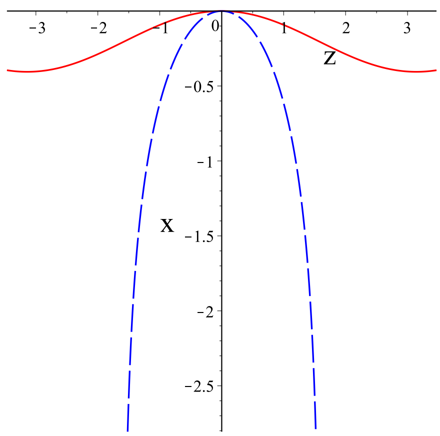


Figure 3.7: Trajectory of a charged particle in a planar undulator, red trajectory is $L = 0.2$ and blue is $L = 0.999$.

Figure 3.7 above shows the motion in the xz plane for different values of the parameter L.

Helical Undulator

A simple extension to the planar undulator is the helical undulator. In this case both the transverse components of A^μ are non-vanishing,

$$A^\mu = (0, \frac{B_0}{k}\sin(kz), \frac{B_0}{k}\cos(kz), 0)\,, \tag{3.70}$$

with the corresponding Hamiltonian,

$$H = \sqrt{m^2 + (P^1 - (B_0/k)\sin(kz))^2 + (P^2 - (B_0/k)\cos(kz))^2 + P_3^2}\,. \tag{3.71}$$

As with the planar undulator, the Hamiltonian, P^1 and P^2 are conserved, so the system is integrable. We now follow [27] and show that this system is maximally superintegrable. The helical undulator also possesses a fourth Poincaré symmetry, the helical generator,

$$L_z = P^3 - k\,M^{12}\,. \tag{3.72}$$

Although this conserved quantity is not in involution with the conserved transverse components of momenta, the helical undulator is minimally superintegrable. These four conserved quantities are the relativistic interpretations of the integrals found in the non-relativistic limit [31]. We are also able to describe the fifth integral by way of the equations of motion for x, z and P^3,

$$\begin{aligned} x' &= -\frac{\partial H}{\partial P^1} = -\frac{P^1 - (B_0/k)\sin(kx)}{H}\,, \\ z' &= -\frac{\partial H}{\partial P^3} = -\frac{P^3}{H}\,, \\ P^{3\prime} &= \frac{\partial H}{\partial z} = \frac{k}{H}(P^1\cos(kz) - P^2\sin(kx))\,, \end{aligned} \tag{3.73}$$

where the prime denotes the derivative with respect to t. We observe that the only change between the relativistic expressions and the non-relativistic limits lie in the inclusion of an additional factor of H, which is conserved, in the denominator. We amalgamate the equations in (3.73) to find,

$$\frac{dx}{P^1 - (B_0/k)\sin(kz)} = \frac{dz}{P^3} = \frac{dz}{\sqrt{2(B_0/k)(P^2\cos(kz) + P^1\sin(kz) + u)}}\,, \tag{3.74}$$

where $u = H^2 - P_1^2 - P_2^2 - 1 - (B_0/k)^2 = const$. This suggests that there is a fifth conserved quantity, however it is non-polynomial in the canonical momenta, as seen in the non-relativistic limit [31].

This represents an undulator of infinite length, the canonical momentum once again the most convenient starting point to investigate the individual components of momenta. From these we are able to then solve the equations of motion. Again, it is still possible to integrate directly for $\mu = 0, 1, 2$ and we find the results to be

$$\begin{aligned} p_0 &= const = \gamma_0 mc \\ p_x &= p_{x,0} - \frac{e}{c}[A_x(z) - A_x(z_0)] \\ p_y &= p_{y,0} - \frac{e}{c}[A_y(z) - A_y(z_0)] \end{aligned} \tag{3.75}$$

We note that now the mechanical momentum is no longer conserved in the y direction as it was previously. As for the previous case, the mass shell relation is required to solve for the momentum in the z direction. So we have that

$$p_z^2 = p_0^2 - m^2c^2 - 4(B_0/k)^2(1 - \cos(kz - kz_0)). \tag{3.76}$$

If we define $kz - kz_0 = \Delta z$, then we can express the z momentum as

$$p_z = \sqrt{p_0^2 - m^2c^2 - 8(B_0/k)^2 \sin^2(\Delta z/2)}. \tag{3.77}$$

We find that this has a similar form to the simpler, linear case but the argument of the sin function is now halved.

If it is assumed that the particle is initially at the origin for this case, then we have

$$p_z = \sqrt{p_0^2 - m^2c^2 - 8(B_0/k)^2 \sin^2(kz/2)}. \tag{3.78}$$

Is it possible to equate the constant part of the square root with $p_{z,0}$ from the first formulation. So it becomes a similar expression to the simpler linear case previously but with half the angle in the argument. So we would get Jacobian sine and cosine functions for the x and y directions.

3.4 Conclusion

We have shown that a particle moving in a vector background field can possess the symmetries of the Poincaré group. We began by investigating electromagnetic fields that are constant in both space and time. These cases can be split into four distinct cases, categorised by the motion they exhibit. We showed that the motion of a particle in a constant magnetic, electric or crossed field is maximally superintegrable. Although the constant magnetic field and constant electric field had different canonical momenta conserved, both fields had the symmetries of the rotation about z and the boost along z. For the magnetic field the rotation required a modification and the boost required a modification in the electric field case. In the case of the crossed field, we showed that the orbits may be obtained algebraically from the conserved quantities. The combination of magnetic and electric fields does not appear to be superintegrable, as, although there exist four conserved quantities, no three of them are in involution. The final examples that we investigated were fields that were all of the form $F^{\mu\nu} = F^{\mu\nu}(l.x)$. These fields only depended on one spacetime coordinate. The first example of the plane wave was shown to be maximally superintegrable and shared the same conserved quantities as the constant crossed field, which is to be expected as the crossed field is the low frequency limit of the plane wave. The addition of a longitudinal constant electric resulted in the loss of the two null rotations and hence was only integrable. However, we were still able to express the orbits in terms of known functions. The final example, where l was space-like was the undulator. The planar undulator is minimally superintegrable as in addition to the three conserved components of momenta, a modified rotation about z was also conserved. The helical undulator was shown to be maximally superintegrable and included a symmetry that was non-polynomial in the momenta that does not seem to be present for the planar undulator.

Chapter 4

Scalar Fields

In this chapter we will discuss the action and equations of motion for a relativistic particle in a scalar background field. We will once again show that a symmetry of the background field implies a conserved quantity in the particle motion and we will also show that the scalar field has an enlarged symmetry group that includes dilations and conformal symmetry in addition to the Poincaré symmetries.

4.1 Introduction

The vector structure of the field $A^\mu(x)$ imposes strict constraints which can hinder the generation of conserved quantities, so we therefore turn our attention to the case of a particle moving in a scalar background field. The scalar field differs from the vector field in that at every point in space-time there is only an associated magnitude, there is no associated direction. Scalar fields are required to be independent of the choice of reference frame, therefore observers will agree on the value of the scalar field at the same absolute point in spacetime, regardless of their respective points of origin. This feature of scalar fields is due to scalar fields being invariant under Lorentz transformations. Mathematically, for a scalar field we have $\phi'(x) = \phi(\Lambda^{-1}x)$ under Lorentz transformation Λ and for a vector field, $A'^{\mu}(x) = \Lambda^{\mu}{}_{\nu} A^{\nu}(\Lambda^{-1}x)$. We see that there is an additional Λ in front for the vector field. A scalar field, for instance, can be used to describe the potential energy associated with a particular force, examples of this include the electric potential in electrostatics or the Newtonian gravitational potential. The force, a vector field, can then be obtained from taking the gradient of the potential energy scalar field.

A particle moving in a scalar background field has been used as an early model of gravity [32, 33]. In quantum field theories, the scalar field is associated with spin-0 particles [34], with the Higgs field as the only fundamental scalar field that has been observed in nature [35, 36]. However, scalar quantum fields feature in effective field theory descriptions of certain phenomena, an example of this is the pion which is actually a pseudoscalar [37, 38].

4.1.1 Averaged Motion

We shall now consider the averaged motion of a particle moving in an oscillatory field as a means to get from a vector to scalar background field.

We recall that the equation of motion, written in terms of the canonical momentum, is

$$\dot{P}^\mu = \frac{d}{d\tau}(m\dot{x}^\mu + eA^\mu) = e\,\dot{x}^\nu\partial^\mu A_\nu\,. \tag{4.1}$$

To analyse the average motion, we first decompose the particle trajectory into slow and fast components,

$$x^\mu(\tau) = X^\mu(\tau) + \Upsilon^\mu(\tau)\,. \tag{4.2}$$

$X^\mu(\tau)$ denotes the slow components and $\Upsilon^\mu(\tau)$ the fast. We can think of X as a suitably time averaged motion around which we have a fluctuating perturbation Υ. We substitute (4.2) into (4.1), such that the equation of motion becomes,

$$\frac{d}{d\tau}(m\,\dot{X}_\mu + m\,\dot{\Upsilon}_\mu + eA_\mu) = e\,\dot{X}^\nu\partial_\mu A_\nu + e\dot{\Upsilon}^\nu\,\partial_\mu A_\nu\,. \tag{4.3}$$

We first isolate the equation that governs the slow dynamics, $X^\mu(\tau)$. From this we recover the equation of motion for free motion,

$$m\ddot{X}^\mu = 0 \to X^\mu = X_0^\mu + U_0^\mu\tau\,, \tag{4.4}$$

where X_0 and U_0 are the initial position and velocity respectively. The equation of motion for the remaining fast dynamics is then,

$$\frac{d}{d\tau}(m\dot{\Upsilon}_\mu + eA_\mu) = e\,U_0^\nu\partial_\mu A_\nu(X) + e\dot{\Upsilon}^\nu\partial_\mu A_\nu(X)\,. \tag{4.5}$$

By expanding the derivative on the left hand side, we are then able to rearrange the expression. We make use of the relationship between the gauge and field strength tensor, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, to give,

$$m\ddot{\Upsilon}_\mu = e\, U_0^\nu F_{\mu\nu}(X) + ... \tag{4.6}$$

where the ellipsis denote the higher terms. We are interested in the time averaged motion in an oscillatory field, which is found to be

$$m\langle\ddot{\Upsilon}_\mu\rangle = e\, U_0^\nu \langle F_{\mu\nu}(X)\rangle = 0\,, \tag{4.7}$$

as the average of $F^{\mu\nu}$ is zero and the angular brackets denote the time average. We now follow [39] and adopt the gauge $U_0.A = 0$. We are then able to reconstruct, to the first order, the usual result that the canonical momentum is constant,

$$\frac{d}{d\tau}(m\dot{\Upsilon}_\mu + eA_\mu(X)) \equiv \dot{P}^\mu_{(1)} = e\partial_\mu(U_0.A) = 0\,. \tag{4.8}$$

Integrating this provides us with an expression for the canonical momentum, which we rearrange to recover equation (5) in [39],

$$m\dot{\Upsilon}^\mu = m\, U_0^\mu - eA^\mu(X)\,. \tag{4.9}$$

By also including the constraint that $\langle A(X)\rangle = 0$, the average of the fast dynamics is, to first order,

$$\langle\dot{\Upsilon}^\mu\rangle = U_0^\mu + \frac{e}{m}\langle A^\mu(X)\rangle = U_0^\mu\,. \tag{4.10}$$

We now take our rearranged expression for the canonical momentum (4.9) and substitute into (4.1), and obtain the average, to second order,

$$\langle\dot{P}^\mu\rangle = \langle m\ddot{x}^\mu + e\dot{A}^\mu(X+\Upsilon)\rangle = e\langle\dot{\Upsilon}^\nu\partial^\mu A_\nu(X)\rangle\,. \tag{4.11}$$

Expanding the term on the right hand side and making the assumption that $\langle\dot{A}(X+\Upsilon)\rangle = 0$ so that to the first order we have that the velocity is U_0^μ, we finally obtain the averaged equation of motion to be,

$$\langle\ddot{x}^\mu\rangle = -\frac{e^2}{m^2}\partial^\mu\langle A^\nu(X)A_\nu(X)\rangle \equiv \frac{1}{2}\partial^\mu a_0^2\,. \tag{4.12}$$

a_0^2 is the intensity parameter [40] which is defined as

$$a_0^2 \equiv \langle a^2\rangle = -\frac{e^2}{m^2}\langle A^2\rangle = \frac{e^2}{m^2\Omega^2}U_{0\mu}\langle F^{\mu\alpha}F^\nu_{\ \alpha}\rangle U_{0\nu} \tag{4.13}$$

for the gauge field $A_\mu(x) = Re[a_\mu(x)e^{i\phi(x)}]$ with $\Omega \equiv U_0.\partial\,\phi(X)$.

Alternatively, following [39], let us start with the gauge field,

$$A_\mu(x) = Re[a_\mu(x)e^{i\phi(x)}]\,, \tag{4.14}$$

where the function $\phi(x)$ is real and the complex amplitude function $a_\mu(x)$ varies slowly in space and time. $a_\mu(x)$ also satisfies the gauge condition, $k.a = 0$, where $k_\mu = \partial_\mu \phi$. We may then approximate the time average of gauge to be,

$$a_0^2 = \frac{1}{2}(e/m)^2[-a^*(x).a(x)]\,, \tag{4.15}$$

the $*$ denoting the complex conjugate. As long as the relative variations in the propagation vector and the amplitude function in a period of a few oscillations is small, the function a_0^2 may legitimately be regarded as a scalar function of x that is independent of the constant four-velocity U_0. In general, the norm of the velocity, the time average of the velocity is not equal to one, however, $v^2 = 1$ is always true. Adopting the boundary condition, $\langle v\rangle^2 = 1$ when $a_0^2 = 0$ gives,

$$\langle v\rangle^2 = \langle ds/d\tau\rangle^2 = 1 + a_0^2\,, \tag{4.16}$$

where s denotes the proper time of an observer moving with the averaged velocity of the particle. The observer's velocity is

$$\frac{d\langle x^\mu\rangle}{ds} = \frac{\langle v^\mu\rangle}{\sqrt{\langle v\rangle^2}} = \frac{\langle v^\mu\rangle}{\sqrt{\langle 1 + a_0^2\rangle^2}}\,. \tag{4.17}$$

It is more useful to express these in terms of the particles average momenta,

$$\langle p^\mu\rangle = m\langle v^\mu\rangle = m^*\frac{d\langle x^\mu\rangle}{ds}\,, \tag{4.18}$$

where we have introduced m^* which is the effective mass,

$$\langle p \rangle^2 = m^{*2} = m^2(1 + a_0^2)\,. \tag{4.19}$$

This form of the effective mass was also recovered for the semi-classical plane wave case [41] and quantum-mechanical calculations [42].

For a relativistic particle that has variable rest mass m^*, the equation of motion is given by varying the following action,

$$S = -\int ds\, m^*(x)\sqrt{\dot{x}^\mu \dot{x}_\mu}\,. \tag{4.20}$$

The parameter s is an arbitrary parameter, not necessarily proper time. By imposing the constant $\dot{x}^2 = 1$, which is consistent provided that $m^* \neq 0$, the equation of motion takes the form,

$$\frac{d}{ds}(m^*(x)\dot{x}_\mu) = \partial_\mu m^*(x)\,. \tag{4.21}$$

Equivalently, using equations (4.16) and (4.18), the equation of motion (4.12) may be written,

$$\frac{d}{ds}\langle p_\mu \rangle = \frac{m\langle \dot{v}_\mu \rangle}{\langle ds/d\tau \rangle} = \frac{\frac{1}{2}m\,\partial_\mu a_0^2}{\sqrt{1 + a_0^2}} = \partial_\mu m^*\,. \tag{4.22}$$

The consistency between equations (4.21) and (4.22) reveals that the averaged motion of the particle is exactly that of a classical relativistic particle that has variable rest mass m^*.

4.1.2 Dynamics of a particle in a scalar field

We define the scalar field $m(x)$ to be

$$m(x) \equiv m_0 + V(x)\,. \tag{4.23}$$

We see that the scalar background field couples to the particle like a mass and due to this we shall we refer to $m(x)$ as a dynamical mass from here on. If we compare this to the effective mass defined in (4.19), we see that the function $V(x) = m_0\, a_0^2$, a "ponderomotive potential". For a recent investigation of non-relativistic superintegrable systems with dynamical mass see [43]. From here we shall now follow [44].

The action of a relativistic particle that has rest mass m_0 in a scalar background field $V(x)$ is, see e.g. [45],

$$S = -\int d\tau\, m(x)\sqrt{\dot{x}^\mu \dot{x}_\mu}\,, \tag{4.24}$$

where $x^\mu \equiv x^\mu(\tau)$, where τ is the proper time parameterising the worldline and the dot denotes the derivative with respect to the proper time. This is the same as (4.20) with the effective mass m^* replaced by the scalar field $m(x)$.

We vary the action (4.24) to get the Euler-Lagrange equations,

$$\frac{d}{d\tau}(m\dot{x}^{\mu}) = \partial_{\mu} m \,, \tag{4.25}$$

from these equations we find that $\dot{x}^2$ = constant and therefore the particle is on-shell, we shall fix $\dot{x}^2 = 1$ from here. Equations (4.25) may be regarded as a force law,

$$m\ddot{x}_{\mu} = (g_{\mu\nu} - \dot{x}_{\mu}\dot{x}_{\nu})\partial^{\nu} m \,, \tag{4.26}$$

where the expression on the right hand side replaces the Lorentz force in (3.6) and the tensor structure guarantees the orthogonality of velocity and acceleration, $\dot{x} \cdot \ddot{x} = 0$, and so the constancy of $\dot{x}^2$.

As we did in the vector case, we want to show that a symmetry of the background field, or dynamical mass, implies that there is a conserved quantity in the motion of the particle.

For the scalar background field we find that from (4.25), the canonical momentum P^{μ} is

$$P_{\nu} = m(x)\dot{x}_{\nu} \,. \tag{4.27}$$

In contrast to the canonical momentum for a vector field, where the gauge field appeared as an additional term, here the scalar fields interacts directly with the mechanical momentum $\dot{x}_{\nu}$.

As before we define $\xi^{\mu}(x)$ to be a vector field that defines the infinitesimal form of some coordinate transformation and a conserved quantity as $Q = \xi \cdot P$.

From the equations of motion (4.25), we can show that

$$2m(x)\frac{dQ}{d\tau} = \mathcal{L}_\xi m^2 + P^\mu P^\nu(\partial_\mu\xi_\nu + \partial_\nu\xi_\mu)\,, \tag{4.28}$$

where $\mathcal{L}_\xi = \xi\cdot\partial$, is the Lie derivative of a scalar quantity. If Q is a conserved quantity, then the right hand side of (4.28) must vanish due to the properties of the field and the transformation, not due to the orbit. The equation (4.28) has essentially the same right hand side as the vector case (3.8), the difference being an extra power of P in the Lie derivative term. In the scalar case, to kill the right hand side, the most general method is to contract $P^\mu P^\nu$ with the metric tensor so we can replace it with $m^2(x)$ and therefore only the sum of the two terms needs to vanish. So ξ must obey

$$\partial_\mu\xi_\nu + \partial_\nu\xi_\mu \propto \eta_{\mu\nu} \implies \partial_\mu\xi_\nu + \partial_\nu\xi_\mu = \frac{1}{2}\eta_{\mu\nu}\partial\cdot\xi\,. \tag{4.29}$$

This is just the conformal Killing equation that has the 15 parameter solution

$$\xi_\mu(x) = a_\mu + \omega_{\mu\nu}x^\nu + \lambda x_\mu + c_\mu x^2 - 2(c\cdot x)x_\mu\,, \tag{4.30}$$

which describes translations, Lorentz transformations, dilations and special conformal transformations respectively. Therefore $\xi\cdot P$ is a conserved quantity in the particle motion when the dynamical mass obeys

$$\mathcal{L}_\xi m^2 + m^2\frac{1}{2}\partial\cdot\xi = 0\,. \tag{4.31}$$

The dynamical mass must be symmetric for translations and Lorentz transformations and must transform with a weight for dilations and special conformal transformations.

4.1.3 Hamiltonian formulation

As previously with vector fields, the Hamiltonian that corresponds to (4.24) vanishes due to Euler's homogeneous function theorem. Once again, we solve this problem by selecting a time parameter from one of the physical coordinates of x^μ [12]. Due to relativistic covariance, there is no unique choice, however a particular choice may be more advantageous in particular situations. Each choice has its own set of six phase space variables and a Hamiltonian that is given by a particular component of the canonical momenta that is found by rearranging the dynamical mass-shell constraint $p.p = m^2(x)$. Again, here we shall require the "instant form" and "front form", for derivations and references see [27] and the review [13].

In "instant form", we take the coordinate t as our time parameter and the six-dimensional phase space is then spanned by the coordinates $\mathbf{x} = (x^i) = (x, y, z)$ and their conjugate momenta, $\mathbf{P} = (P^i) = (P^1, P^2, P^3)$. The Hamiltonian is

$$H \equiv P^0 = \sqrt{\mathbf{P} + m^2(t, \mathbf{x})}\,, \tag{4.32}$$

which may be explicitly time-dependent due to m^2. The time evolution of any quantity Q is determined by

$$\frac{dQ}{dt} = \frac{\partial Q}{\partial t} + \{Q, H\}, \tag{4.33}$$

where the Poisson bracket of two phase space functions is

$$\{X, Y\} = \frac{\partial X}{\partial x^i}\frac{\partial Y}{\partial P_i} - \frac{\partial X}{\partial P_i}\frac{\partial Y}{\partial x^i}\,. \tag{4.34}$$

In "front form", we take the coordinate $x^+ \equiv t + z$ as our time parameter and the six-dimensional phase space is then spanned by the 'longitudinal' coordinate $x^- \equiv t - z$, the 'transverse' coordinates $\mathbf{x}^\perp = (x^\perp) = (x, y)$ and their conjugate momenta, P^+ and $\mathbf{P} = (P^\perp) = (P^1, P^2)$. The Hamiltonian is and Poisson bracket are,

$$H \equiv P^- = \frac{\mathbf{P}^\perp\mathbf{P}^\perp + m^2(x^\mu)}{P^+}\,, \tag{4.35}$$

$$\{X, Y\} = -2\left(\frac{\partial X}{\partial x^-}\frac{\partial Y}{\partial P_+}\right) - \frac{\partial X}{\partial P_+}\frac{\partial Y}{\partial x^-} + \frac{\partial X}{\partial x^\perp}\frac{\partial Y}{\partial P_\perp} - \frac{\partial X}{\partial P_\perp}\frac{\partial Y}{\partial x^\perp}\,, \tag{4.36}$$

where the summation convention is used through out for the index $\perp$. The time evolution of any quantity Q is determined by

$$\frac{dQ}{dx^+} = \frac{\partial Q}{\partial x^+} - \{Q, H\}. \tag{4.37}$$

4.2 Simple Examples

We wish to first consider scalar fields that correspond to the three choices of the vector n^μ in the vector case. There are only three individual choices for the vector n, these are that the vector is space-like, time-like or light-like.

4.2.1 The space-like case

We first consider the case where the vector n^μ is space-like, this case can be thought of as the scalar analogy of the position-dependent magnetic field case [27, 31] due to both fields only depending on a space-like vector. In this case the scalar field is defined as a function of one spatial coordinate, say $n.x = z$, and we consider the simple choice

$$m^2(x) = m_0^2 + Bz\,, \tag{4.38}$$

linear in z. We can think of the constant B as the magnetic field strength. The Hamiltonian for this system is then defined as

$$H = \sqrt{\mathbf{P}^2 + m_0^2 + Bz}\,. \tag{4.39}$$

The system is autonomous and therefore the Hamiltonian is a conserved quantity in addition to two of the spatial momenta P^1 and P^2. To find further conserved quantities that do not relate to Poincaré symmetries, we make the following ansatz:

$$Q = f_1(x, y, z, P^3)P^1 + f_2(x, y, z, P^3)P^2 + f_3(x, y, z, P^3)\,, \tag{4.40}$$

and when calculating the derivative with respect to time, we demand that each term vanishes by solving the resulting series of algebraic or differential equations that arise from equating the powers of momentum to determine the functions f_1, f_2 and f_3. Following this method, we find that for the magnetic field with a linear dependence on z, a linear ansatz is sufficient and the system has the following two conserved quantities that do not relate to Poincaré symmetries, but are 'hidden symmetries' on phase space,

$$Q_x = 2P^1P^3 + Bx \qquad Q_y = 2P^2P^3 + By\,. \tag{4.41}$$

The rotation, M^{12}, is also conserved. However, the rotation can be expressed as a linear combination of the two conserved momenta and the quantities Q_x and Q_y and therefore is not an independent conserved quantity. Our set of five independent conserved quantities is given by, $\{H, P^1, P^2, Q_x, Q_y\}$, where $\{H, P^1, P^2\}$ are in involution which gives integrability. For the system to be maximally superintegrable, the conserved quantities need to be independent. To verify this we define $\mathcal{F} = (Q_1, ..., Q_5)$ and following [15], our five conserved quantities are functionally independent if the 5×6 matrix

$$\mathcal{M} = \left(\frac{\partial \mathcal{F}_l}{\partial x^a}, \frac{\partial \mathcal{F}_l}{\partial P^a} \right) , \tag{4.42}$$

for $a \in \{1, 2, 3\}$, has rank 5. The set of conserved quantities that we shall use to calculate $\mathcal{M}$ is $\mathcal{F} = (Q_x/B, Q_y/B, H/B, P^1, P^2)$, then

$$\mathcal{M} = \begin{pmatrix} 1 & 0 & 0 & \frac{2P^3}{B} & 0 & \frac{2P^1}{B} \\ 0 & 1 & 0 & 0 & \frac{2P^3}{B} & \frac{2P^2}{B} \\ 0 & 0 & 1 & \frac{2P^1}{B} & \frac{2P^2}{B} & \frac{2P^3}{B} \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} , \tag{4.43}$$

which is upper triangular with rank 5 and therefore the system is maximally superintegrable.

We now solve the Hamiltonian equations for the particle trajectory and we find that we have trivial dynamics in the x and y direction due to the conservation of the respective momenta. The canonical momentum in z direction is

$$P^z(t) = \frac{Bt}{2H} + P^z_0 \, . \tag{4.44}$$

To find the trajectory in the z direction we use the conservation of the Hamiltonian and rearrange this to give

$$z(t) = \frac{H^2 - (P^1)^2 - (P^2)^2 - m_0^2 - (P^3)^2}{B} \, . \tag{4.45}$$

To have a physical 'scattering' boundary conditions, we consider the case where the field 'switches on'. The dynamical mass is redefined to obey

$$m^2(t,\mathbf{x}) = \begin{cases} m_0^2 & z < 0 \\ m_0^2 + Bz & z \geq 0 \end{cases}, \tag{4.46}$$

and we consider the motion of particles which reaches the interface $z = 0$ at $t = 0$, without any loss of generality. We are able to specify the initial momentum at $t \leq 0$ as the motion is free at $t < 0$. The initial momentum at $t = 0$ then fixes the values of the conserved quantities and P_0^z.

Figure 4.1: Trajectories in the autonomous system (4.39) with dynamical mass $m^2 = m_0^2 + Bz$ with $P_\perp = 0$ and $m_0 = B = 1$.

Figure 4.1 shows the trajectory of the particle entering the field (shaded) at $\mathbf{x} = 0$ at time $t = 0$. The trajectory shows the particle moving through the field to a certain distance in the region $z > 0$ before it is then turned around and pushed out of the field.

4.2.2 The time-like case

The next case we shall consider is the case where the vector n^μ is taken to be time-like. In the previous chapter we saw that an electric field was generated by a gauge which was time dependent, $\mathbf{A} = -\mathbf{E}t$. The entries of the electromagnetic fields tensor, $F^{\mu\nu}$, are then given by $-E^i = F^{0i} = \partial^0 A^i - \partial^i A^0$. If we replace the gauge field $A^\mu(t)$ with the function $\phi(t)$, we can consider the time-like case as the analogue to the vector electric field. We choose $n.x = t$ and the scalar field is then

$$m^2(x) = m_0^2 + E(t) \tag{4.47}$$

for $t \geq 0$. The Hamiltonian for this system is

$$H(t) = \sqrt{\mathbf{P}^2 + m_0^2 + E(t)}, \tag{4.48}$$

and we can see here that in this case the Hamiltonian depends explicitly on the time and is therefore not conserved. However as the Hamiltonian depends only on the chosen time coordinate, all three spatial momenta are conserved. In addition to these, the three components of angular momenta are also conserved. The set of five conserved quantities is then the three conserved momenta, that are in involution and two of the three conserved components angular momentum. Due to the angular momenta, L_i, obeying the relationship $P_i L_i = 0$, this implies that not all three components of angular momentum are independent, which renders the system maximally superintegrable. Owing to the conservation of the three spatial momenta, the equations of motion for the particle trajectories all take the same form,

$$\frac{dx^i}{dt} = -\frac{P^i}{H(t)}\,, \qquad i = 1, 2, 3\,, \tag{4.49}$$

which have the trivial solution,

$$x^i(t) = x_0^i - \int_{t_0}^{t} ds \frac{P_0^i}{\sqrt{\mathbf{P}^2 + m_0^2 + E(s)}}\,, \qquad i = 1, 2, 3\,. \tag{4.50}$$

These equations give an implicit solution where the choice of the function $E(s)$ will determine whether the integration can be performed analytically. In principal, as all three components of angular momentum are conserved, only the one integral needs to be performed and from this the coordinates for the remaining directions can then be found algebraically, as can be seen from (4.50). To illustrate, take the simplest case where the the function $E(t)$ is linear in t, so that the Hamiltonian is

$$H(t) = \sqrt{\mathbf{P}^2 + m_0^2 - Et}, \tag{4.51}$$

where we can think of the constant E as the electric field strength. The particle trajectories are then

$$
\begin{aligned}
x(t) &= x(0) + \frac{2P_1\sqrt{\mathbf{P}^2 + m_0^2 - Et}}{E}\,, \\
y(t) &= y(0) + \frac{2P_2\sqrt{\mathbf{P}^2 + m_0^2 - Et}}{E}\,, \\
z(t) &= z(0) + \frac{2\,P_3\sqrt{\mathbf{P}^2 + m_0^2 - Et}}{E}\,.
\end{aligned}
\tag{4.52}
$$

The trajectory of a particle in the tz plane is shown in Figure 4.2 below.

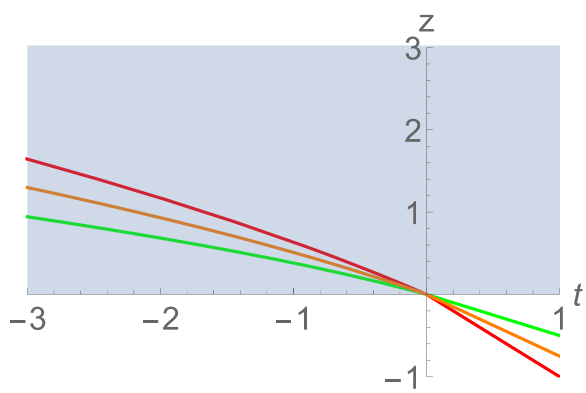


Figure 4.2: Trajectories for time-like system (4.48) with dynamical mass $m^2 = m_0^2 - Et$ with $P_\perp = 0$ and $E = m_0^2 = 1$.

The field is switched on for $z = 0$ and we consider the motion of particles that reach the interface $z = 0$ at $t = 0$, without any loss of generality. Here, the particle starts in the field where it decelerates and leaves the field after which it then experiences free motion. This appears to have the opposite effect to its vector analogy where we saw that the field accelerated the particle.

In autonomous Hamiltonian mechanics, the Hamiltonian itself is one of the conserved quantities. However, in this case we see that the Hamiltonian depends on the time parameter explicitly and the system is non-autonomous. We can create an autonomous system by enlarging the phase space to eight dimensions. We now require the system to have seven conserved quantities for the system to be maximally superintegrable. In this enlarged phase space, our time parameter t is an additional coordinates with its conjugate momenta P^0 and our new Hamiltonian is define as $K = H - P^0$ [46]. The time-derivative of any quantity Q is now

$$Q' = -\{Q, K\}_* \quad \text{where} \quad \{X, Y\}_* = \frac{\partial X}{\partial x^\mu}\frac{\partial Y}{\partial P_\mu} - \frac{\partial X}{\partial P_\mu}\frac{\partial Y}{\partial x^\mu}\ , \mu \in \{0, 1, 2, 3\}\,. \tag{4.53}$$

The new time now does not appear explicitly and $t' = -\partial K/\partial P^0 = 1$. We can verify that in addition to the five conserved quantities found in the non-autonomous system, we have an additional two, one of which is the new Hamiltonian by construction, which is quadratic in the momenta and encodes the dynamical mass-shell constraint. The final conserved quantity is the initial value for one of the spatial coordinates. For example, we could take the set of conserved quantities to be,

$$P^1, \quad P^2, \quad P^3, \quad M^{23}, \quad M^{31}, \quad K, \quad x + P_1\int_{t_0}^{t} ds \frac{1}{\sqrt{\mathbf{P}^2 + m_0^2 + E(s)}} = x_0\,. \tag{4.54}$$

The set of conserved quantities depends on which of the two components of angular momenta we take to be our conserved quantities which determines the direction to take as the seventh conserved quantity due to the constraint on the momentum $\mathbf{p}.\mathbf{L} = 0$. In total we have seven globally defined and independent conserved quantities that are polynomial in the momenta and the set $\{K, P^1, P^2, P^3\}$ is in involution. Therefore the we have a polynomial maximally superintegrable system [15].

4.2.3 The light-like case

The final case we shall consider is the case where the vector n^μ is taken to be light-like. We consider this as the scalar analog to the plane wave as any field of this form is a solution to the wave equation. For the scalar field, with field strength F, we choose the field to dependent on the choice $n.x$,

$$m^2(x) = m_0^2 + F\, n \cdot x \,. \tag{4.55}$$

In front form, the Hamiltonian is

$$H = \frac{P_\perp^2 + m^2(n.x)}{P^+} \,. \tag{4.56}$$

We have two choices for the variable, either $n.x = x^-$ or $n.x = x^+$, which will create either an autonomous or non-autonomous system.

For an autonomous system we choose $n.x = x^-$, so that the Hamiltonian and the two transverse momenta, P^1 and P^2, are conserved and in involution so we have at least integrability. These conserved quantities correspond to translation invariance in three dimensions. Plane waves are also invariant under null rotations [27, 47], giving the corresponding conserved quantities

$$Q_{\perp} = Hx^{\perp} + x^{-}P^{\perp}\,. \tag{4.57}$$

In field theory applications, it is often more convenient for the dependence of the plane wave to coincide with the choice of time, $\bar{n} \cdot x = x^{+}$. Now the system is non-autonomous as the Hamiltonian depends explicitly on the time parameter, however now all three momenta are conserved and in involution. As we have previously done in the time-like case, we can create an autonomous system by extending the phase space to eight dimensions and create a new Hamiltonian, $K = H - P^{-}$. The time derivative of a quantity Q is given by (4.53). From here we can verify that the five conserved quantities following from the invariance of the plane wave under translations and rotations are

$$Q_1 = P^1, \quad Q_2 = P^2, \quad Q_3 = P^{+}, \quad Q_4 = xP^{+} + x^{+}P^1, \quad Q_5 = yP^{+} + x^{+}P^2\,. \tag{4.58}$$

The additional two conserved quantities involve the extended phase space variables. By construction one of these is the new Hamiltonian,

$$Q_6 = P^{+}P^{-} - P^{\perp}P^{\perp} - m^2(x^{+}), \tag{4.59}$$

which is quadratic in the momenta and encodes the dynamical mass-shell constraint. The final conserved quantity is

$$Q_7 = P^{+^2} x^- - P^\perp P^\perp - \int dx^+ m^2(x^+)\,, \tag{4.60}$$

which, in the original phase space, immediately gives the solution to the equations of motion for x^-. In total we have seven globally defined and independent conserved quantities that are polynomial in the momenta and the set $\{Q_1, Q_2, Q_3, Q_6\}$ is in involution. Therefore the we have a polynomial maximally superintegrable system [15]. The solution to the equations of motion are as follows. The solutions to the transverse directions are given by the two null rotations,

$$\begin{aligned} x(x^+) &= \frac{Q_4 - x^+ P^1}{P^+}\,, \\ y(x^+) &= \frac{Q_5 - x^+ P^2}{P^+}\,, \end{aligned} \tag{4.61}$$

and the final direction is given by Q_7.

To illustrate, we shall consider two choices of background field, one that is sinusoidal and models a monochromatic plane wave and the second that is linear in x^+ which models the low frequency limit of the plane wave. For the sinusoidal choice of background field, the dynamical mass is redefined to obey

$$m^2(t, \mathbf{x}) = \begin{cases} m_0^2 & x^+ < 0 \\ m_0^2 + F\sin(\omega x^+) & x^+ \geq 0 \end{cases}, \tag{4.62}$$

and consider the motion of particles which reach the interface $x^- = 0$ at $x^+ = 0$, without any loss of generality. The dynamics in the non-trivial plane are given by

$$x^-(x^+) = \frac{Q_7 + m_0^2 x^+ - (F/\omega)\cos(\omega x^+)}{{P^+}^2}. \tag{4.63}$$

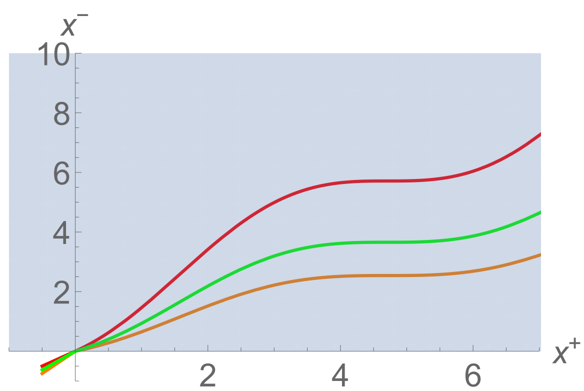


Figure 4.3: Trajectories for dynamical mass $m_0^2 + \sin(x^+/\omega)$ with $P_\perp = 0$, $P_0 \in \{1, 1.25, 1.5\}$, $Q_7 = F/\omega$ and $m_0 = \omega = F = 1$.

From Figure 4.3 we see that the particle experiences free motion until it enters the field where the behavior then becomes sinusoidal and oscillates in the field. The strength of the oscillation appears to be inversely proportional to the initial momentum. The greater the initial momentum is we find that the oscillations are less pronounced.

In the low frequency limit, the dynamical mass is redefined to obey

$$m^2(t,\mathbf{x}) = \begin{cases} m_0^2 & x^- < 0 \\ m_0^2 + Fx^+ & x^- \geq 0 \end{cases}, \tag{4.64}$$

and again, we consider the motion of particles which reach the interface $x^- = 0$ at $x^+ = 0$. The dynamics in the non-trivial plane are now given by

$$x^-(x^+) = \frac{Q_7 + m_0^2 x^+ + (F/2){x^+}^2}{{P^+}^2}. \tag{4.65}$$

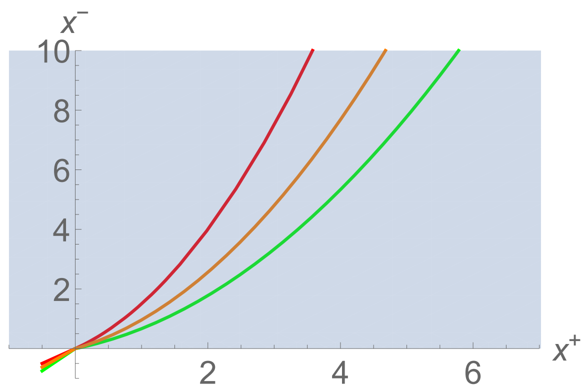


Figure 4.4: Trajectories for dynamical mass $m_0^2 + Fx^+$ with $P_\perp = Q_7 = 0$, $P_0 \in \{1, 1.25, 1.5\}$ and $m_0 = F = 1$.

As shown in Figure 4.4, the particle motion is free until it enters the field where the motion becomes parabolic. Interestingly, we recall that the motion of a particle in a constant null field, the vector equivalent of the low frequency limit, was also parabolic in a particular plane.

4.2.4 Boost invariant example

The previous examples have all been examples where the scalar field depends on a single coordinate. We shall now consider a scalar field that depends on a combination of both light-front coordinates. We define the scalar field to be

$$m^2(x) = \phi(x^+x^-)\,. \tag{4.66}$$

This system is invariant under translations in the transverse directions, thus P^1 and P^2 are conserved. For a particle moving in any background field of this form, we also find the the rotation M^{12} and the boost M^{+-} are also conserved. The set $\{P^1, P^2, M^{+-}\}$ are in involution and all four quantities are functionally independent so we have minimal superintegrability.

For a field of this form, the equations of motion are solved with the aid of the conserved quantities. We write the boost as M_0^{+-}; using these we then get an expression for P^+ in terms of the field variable,

$$P^+(x^+x^-) = -\frac{1}{2x^-}\left(M_0^{+-} \pm \sqrt{(M_0^{+-})^2 + 4x^+x^-(P_{\perp 0}^2 + m^2)}\right)\,. \tag{4.67}$$

To simplify this expression, we make a special choice of m^2,

$$m^2(x^+x^-) = \frac{P_{\perp 0}^4 x^+x^-}{(M_0^{+-})^2}\,, \tag{4.68}$$

which is linear in our choice of field variable x^+x^-. This choice completes the square in (4.67), however it is restrictive as this only works for a particular choice of initial conditions. We also use the constraint (4.68) to simplify the Hamiltonian equation for x^-,

$$\frac{dx^-}{dx^+} = \frac{x^-}{x^+} + \frac{(M_0^{+-})^2}{P_{\perp 0}^2\, x^{+2}}\,. \tag{4.69}$$

This can now be integrated and yields the trajectory for x^-,

$$x^-(x^+) = \frac{x_0^-}{x_0^+}\, x^+ + \frac{(M_0^{+-})^2}{2P_{\perp 0}^2}\left(\frac{x^+}{x_0^{+2}} - \frac{1}{x^+}\right)\,. \tag{4.70}$$

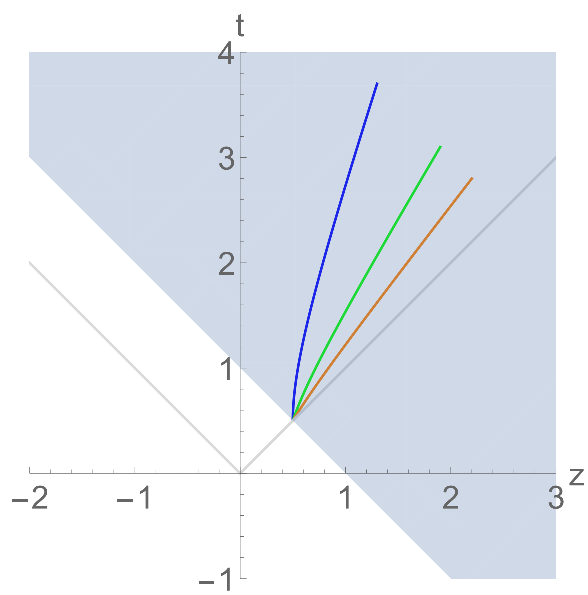


Figure 4.5: Trajectories for dynamical mass $m^2(x^+x^-) = P_{\perp 0}^4 x^+ x^- / (M_0^{+-})^2$ with the lightcone shown in grey, $x_0^- = 0$, $(M_0^{+-})^2/P_{\perp 0}^2 = \{1, 0.5, 0.25\}$

Figure 4.5 shows that the particle enters the background field at the time $x^+ = x_0^+$, and that the field initially accelerates the particle. After the initial acceleration the particle motion then becomes linear.

The constraint on m^2 also means that the expression for P^+ simplifies to

$$P^+ = \frac{P_0^+\, x^+}{x_0^+} \quad \text{where} \quad P_0^+ = \frac{x_0^+\, P_{\perp 0}^2}{M_0^{+-}}\,, \tag{4.71}$$

where the right hand side no longer depends on the product x^+x^-, but only on the time coordinate x^+. The Hamiltonian can then be expressed

$$P^- = \frac{P_{\perp 0} x_0^+}{P_0^+\, x^+} + \frac{P_{\perp 0}^4 x_0^+ x^-}{P_0^+ (M_0^{+-})^2}\,. \tag{4.72}$$

The transverse trajectories are

$$x^\perp(x^+) = x_0^\perp + \frac{P_0^\perp}{P_{\perp 0}^2} \ln\left(\frac{x^+}{x_0^+}\right)\,. \tag{4.73}$$

4.3 Beyond Poincaré

The advantage of working in a scalar field background is that the group of symmetries is extended to include dilations and special conformal symmetries. We shall now present examples of fields that have these additional symmetries.

4.3.1 Dilations

The first example we shall investigate is one that involves dilation symmetry. Dilations with ξ_D^μ yields a conserved quantity $\xi_D \cdot P$ when $m^2(x)$ obeys the following,

$$x \cdot \partial\, m^2(x) = -2m^2(x)\,, \tag{4.74}$$

thus $m^2(x)$ is an eigenfunction of the dilation operator $x.\partial$ with eigenvalue or "weight" 2. The symmetry is generated by

$$D = -ix^\mu \partial_\mu\,, \tag{4.75}$$

where the scale transformations are given by the exponentiation of

$$\exp(i\,\lambda\, D)\phi(x) = \phi(\lambda\, x), \tag{4.76}$$

so that $x^\mu \to \lambda x^\mu$, where λ is the scale factor. Equivalently, we can write this in terms of the "unperturbed" mass m_0, and the scalar background $\phi(x)$, so we have the relationship,

$$x \cdot \partial\, \phi(x) = m_0 - \phi(x)\,. \tag{4.77}$$

To illustrate, we choose an example which has full Lorentz invariance such as $m^2 \equiv m^2(x \cdot x)$, then equation (4.74) fixes the function to have the form

$$m^2 = \frac{C^2}{x \cdot x} = \frac{C^2}{x^+x^- - x^2 - y^2}\,, \tag{4.78}$$

where C is a constant. Thus, including a dilation symmetry and insisting on Lorentz invariance limits the possible choices for the function m^2 to a single choice (4.78). We can show that this choice has four Poincaré symmetries, two null rotations, a rotation about the z axis and a boost along z. The fifth conserved quantity is

$$Q_5 = x^+ H + x^- P^+ + x P^x + y P^y \,. \tag{4.79}$$

The set of conserved quantities of the two null rotations and the dilation symmetry are in involution and functionally independent as the matrix $\mathcal{M}$ has rank 5, therefore the system is maximally superintegrable.

Interestingly, although this is a maximally superintegrable system, we are only able to make analytical progress to obtain the solutions if we make an additional assumption about at least one of the conserved quantities. We consider the case where $Q_5 = 0$. Thus we have that $x.x = u_0$, a constant, following from this the equation of motion for P^+ can then be solved. We take $u_0 > 0$, and hence $m^2 > 0$ and so $P^+ > 0$, and find

$$P^+ = \frac{\sqrt{4u_0^2 {P_0^-}^2 - C^2 {x^+}^2}}{2u_0} \,. \tag{4.80}$$

Using the conditions that $Q_i =$ constant, $i \in \{1, .., 5\}$ as a system of five algebraic equations, we find the remaining five momenta and coordinates, $\{P_\perp, x^\perp, x^-\}$. By eliminating four of the variables from the system this leaves a quadratic equation determining the remaining variables. However, these expressions are complicated and unrevealing.

4.3.2 Special conformal transformations

In addition to dilation symmetries, scalar fields can also possess special conformal symmetry. If we take m^2 to be of the form

$$m^2(x) = \frac{1}{x^{+2}}\, f\left(x^- - \frac{x^\perp x^\perp}{x^+}\right) , \tag{4.81}$$

this will by invariant under the special conformal transformation,

$$\xi_c^\mu = c^\mu x^2 - 2x^\mu c.x \,, \tag{4.82}$$

that is generated by $c^- = 1$ and all other components vanish. This transformation is generated by

$$K^\mu = -i(2x^\mu x^\nu \partial_\nu - x^2 \partial^\mu) \,, \tag{4.83}$$

and can be viewed as a translation that is both preceded and followed by an inversion $x^\mu \to x^\mu / x^2$.

A function of the form (4.81) is also invariant under three Poincaré symmetries, two null rotations and a rotation about the z axis. We can once again move to an enlarged phase space where we can identify the following five conserved quantities,

$$Q_\perp = 2P^+ x^\perp + x^+ P^\perp , \quad Q_3 = \xi_c.P , \quad Q_4 = xP^2 - yP^1 , \quad Q_5 = K \,. \tag{4.84}$$

These are independent and the set $\{Q_1, Q_2, Q_3, Q_5\}$, are four quantities in involution. To solve the equations of motion we define $u = x^- - x^\perp x^\perp / x^+$, then using the conservation of the null rotations and the special conformal symmetry, we are able to write P^+ in terms of u,

$$P^+ = -\frac{Q_\perp^2 + f(u)}{4Q_3} \,. \tag{4.85}$$

The conserved quantity Q_3 must be greater than zero. The Hamiltonian equation for the new variable u is

$$\frac{du}{dx^+} = -\frac{Q_3}{P^+ x^{+2}} = \frac{4Q_3^2}{Q_\perp^2 + f(u)} \frac{1}{x^{+2}} \implies \int_{u_0}^{u} ds \frac{Q_\perp^2 + f(s)}{4Q_3^2} = \frac{1}{x_0^+} - \frac{1}{x^+} \,. \tag{4.86}$$

This then gives us an implicit expression for $u \equiv u(x^+)$, with initial conditions that $u = u_0$ when $x^+ = x_0^+$. From this we can identify an expression for P^+, $P^+ \equiv P^+(x^+)$ using (4.85). We next identify the particle motion in the $x^\perp$ directions from the Hamiltonian equations,

$$\frac{dx^\perp}{dx^+} = -\frac{P^\perp}{2P^+} = \frac{x^\perp}{x^+} - Q_\perp 2x^+ P^+ \,, \tag{4.87}$$

that we can integrate. We then use this to write our expression for x^- as $x^- = u + x^\perp x^\perp / x^+$.

As an example, we choose the background field to be an exponential function and the field is defined as,

$$m^2 = \begin{cases} m_0^2 & x^+ < L \\ \frac{m_0^2 L^2}{{x^+}^2} e^{-k^2(x^- - x^\perp x^\perp / x^+)^2} & x^+ \geq L \end{cases}, \tag{4.88}$$

where k is a parameter with units of inverse length. If we set the transverse momenta to zero so there are no transverse dynamics, the non-trivial part of the orbit is

$$\frac{1}{x^+} = 1 - \kappa \, \mathrm{Erf}(x^-) \tag{4.89}$$

with dimensionless variable $\kappa = 2\sqrt{\pi/k}\,(P^-/(m_0 x_0^+))^2$ in terms of the initial P^-.

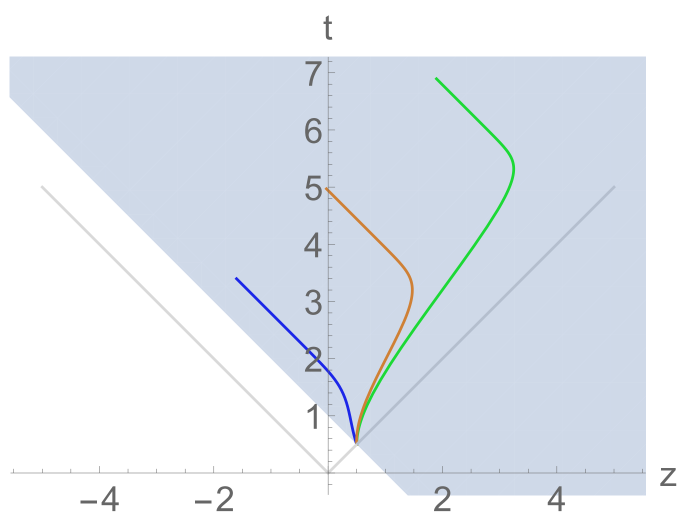


Figure 4.6: Trajectories for dynamical mass $m^2 = (m_0^2 L^2/{x^+}^2) e^{-k^2(x^- - x^\perp x^\perp/x^+)^2}$, with the lightcone shown in grey and $x^\perp = P^\perp = 0$, orbits are plotted for $\kappa = \{0.9, 0.5, 0.3\}$.

As seen in Figure 4.6, the particle enters the background field at the time $x^+ = L$. For larger x^+ the particle approaches the speed of light as the dynamical mass drops to zero. This makes the Hamiltonian equivalent to that of a massless particle. The coordinates x^+ and x^- are measured in units of L and $1/k$ respectively.

4.4 Conclusion

We were able to move from a vector background field to a scalar field through the averaging of the vector dynamics. By considering a particle moving in a scalar background field, we have shown that the symmetry group is extended from 10 parameters to 15. The symmetry group now includes dilations and conformal symmetry in addition to Poincaré symmetries. We began by investigating scalar fields created from the three distinct choices for the vector n and found these all resulted in maximally superintegrable systems. In the cases where the vector n was time-like and light-like, we could extend the phase space to create autonomous systems. We found there to be an additional two conserved quantities in the extended phase space so that these examples remained maximally superintegrable. When we compared these cases to their vector counter parts we find that the null case has the same five conserved quantities in both settings and that the space-like and time-like cases differ from their vector counterparts. We observed that the particle motion in null scalar fields is similar to that seen in the null vector field case and the field has the opposite effect on the particle motion for the time-like case to that found in the constant electric field case. It is interesting to note that although the space-like case is maximally superintegrable, two of the symmetries did not correspond to Poincaré symmetries. We have seen that the scalar plane wave is a maximally superintegrable system and that it has the same conserved quantities as the vector case, however there is no longer a gauge modification required. The boost invariant example was found to be minimally superintegrable. However, to make progress in solving the equation of motion, we had to make a simplification. By fixing the form of m^2, we were able to simplify the expression for P^+, which in turn made it possible to solve for the particle trajectory. We also had to make a restricting assumption in the example which included a dilation symmetry. This example is maximally superintegrable and the trajectories could be obtained algebraically once we assumed that the conserved quantity relating to the dilation symmetry was equal to zero. Our final example was one which included a special conformal symmetry and was found to be only minimally superintegrable. To obtain the particle trajectories we defined a new variable $u = x^- - x^\perp x^\perp / x^+$ and solved the equations of motion in terms of u.

Chapter 5

Summary and Outlook

5.1 Summary

Our aim in this thesis was to determine the trajectory of a relativistic particle moving in a background field. In the classical regime, we solve the Lorentz force equation of motion and ideally we wish for the choice of external field to model a laser.

In Chapter 2, we showed that a symmetry of the background field relates to a conserved quantity in the motion of the charged particle. To solve for the particle trajectory we introduced the concept of integrability. For this, we turned to the Hamiltonian formulation with a phase space of dimensions $2n$. A system is then described as integrable if there exist n independent conserved quantities that are in involution, that is their Poisson brackets vanish. We then extended the concept of integrability to superintegrability, where we now have a further k conserved quantities where $1 \leq k \leq n-1$. A system that has $k=1$ additional symmetries is minimally superintegrable and maximally superintegrable when $k=n-1$.

We first considered examples of a particle in vector background fields described by the gauge field $A^\mu(x)$. In the simplest case, where the background field, $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$, is taken to be constant in space-time, the resulting motion falls into four distinct cases; hyperbolic, elliptic, parabolic and loxodromic. For the first three cases we were able to identify five independent conserved quantities, where three are in involution, and therefore classify these cases as maximally superintegrable. The final case, loxodromic motion, was an interesting one, as although it had four conserved quantities, no three of them are in involution, so it could not be classed as integrable. We then went on to investigate fields that had a space-time dependence on one coordinate only. We showed that the standard model of a laser, the plane wave, was maximally superintegrable. In this case, the orbits can be solved algebraically. When we extended this to include a longitudinal constant electric field, we found that we lost the two null rotations and the system was only integrable. The last example we considered was the undulator. We showed that the planar undulator was minimally superintegrable, with four conserved quantities. Extending this to the helical undulator, the system became maximally superintegrable and included a symmetry that was non-polynomial in the canonical momenta.

We then went on to consider a particle moving in a scalar background field. We began by showing that we can obtain the scalar equation of motion by considering the averaged motion of a particle moving in an oscillatory vector field. We found that by taking the average momentum, this gave rise to an effective mass. The equation of motion for the averaged motion of a particle is exactly that of a classical relativistic particle with a variable rest mass. We then showed that, in a scalar background field, the symmetry group is extended from the 10 parameter Poincaré group to the conformal group of 15 parameters.

The first examples we investigated corresponded to each of the three choices of the vector n^μ. Each of these cases were found to be maximally superintegrable. In the space-like case, the scalar analogy of the position-dependent magnetic field, the field has three conserved quantities that related to Poincaré symmetries. The final two conserved quantities did not relate to Poincaré symmetries and were found by making an ansatz for the form of the conserved quantity and demanding that its time evolution was zero. This was also our only autonomous example. For the time-like and light-like cases, we extended the dimension of phase space to $2n+2$ and defined a new autonomous Hamiltonian K. Interestingly, in the time-like case, there was no unique set of conserved quantities. The set is defined by which of the two conserved angular momenta we decide to take as our conserved quantities. This then fixes the final conserved quantity, an initial position. For the light-like case, we found that the conserved quantities were the same as those found in the vector counter part. The next example we considered was boost invariant. This example was minimally superintegrable with four Poincaré symmetries. In this case, to make progress solving the equation of motion, we had to make a special choice for $m^2(x)$.

Our final examples were those that included a symmetry from the extended symmetry group. The first example included a dilation symmetry in addition to four Poincaré symmetries making it a maximally superintegrable system. To make progress solving the equations of motion, we had to make an assumption regarding one of the conserved quantities. Following from this, we were then able to solve the equation of motion for one of the components of momenta. The remaining coordinates and momenta could then be found algebraically. The second example included a special conformal symmetry as well as three Poincaré symmetries making it minimally superintegrable. We were able to solve the equations of motion with the aid of defining a new variable and solving in terms of this new variable.

5.2 Outlook

5.2.1 Towards the quantum problem

Moving forward, it would be interesting to extend these ideas to a quantum setting, especially in light of the conjecture made by Tempesta [18] that all maximally superintegrable systems are exactly solvable. To study the quantum mechanical analogues of classical superintegrable systems, we replace the Poisson brackets with commutators [31]. As relativistic quantum mechanics is problematic [48], we instead use relativistic quantum field theory. The 'first quantised', quantum mechanical, approach still has a role to play. For scalar fields, the solutions to the Klein-Gordon equation give asymptotic particle wave functions which are used as the basis of scattering amplitudes. These solutions will typically contain physics which cannot be captured by single particle dynamics, such as pair production. Therefore it is not obvious how the superintegrability of a relativistic particle system translates to its field theory analogue.

5.2.2 Radiation reaction

When a charged particle is accelerated it emits electromagnetic energy in the form of electromagnetic waves and the energy the particle loses in this process is proportional to the square of the acceleration according to Larmor's formula. This process produces a self force that acts on the particle that is being accelerated and the force is known as radiation reaction force, or just radiation reaction. Radiation reaction was first discovered by Abraham [49] and his work was extended by Lorentz [50]. It was then re-derived by Dirac [51]. Radiation reaction is then described by the Lorentz- Abraham- Dirac (LAD) equation,

$$\dot{p}^\mu = \frac{e}{m} F^{\mu\nu} p_\nu + \tau_0 \mathbb{P}^{\mu\nu} \ddot{p}_\nu \,, \tag{5.1}$$

where τ_0 is a parameter defined as $\tau_0 = e^2/6\pi\epsilon_0$, with the dimensions of time and $\mathbb{P}^{\mu\nu}$ is a projection perpendicular to p to guarantee $p.\dot{p} = 0$. However, there are two issues with this equation in that it has runaway solutions, these are solutions that diverge exponentially and preacceleration which breaks the rule of causality. An attempt to avoid these issues was made by Landau and Lifshitz by approximating the LAD equation [8] with

$$\dot{p}^{\mu} = \frac{e}{m} F^{\mu\nu} p_{\nu} + \tau_0 \frac{e}{m} \mathbb{P}^{\mu\nu} (\dot{F}_{\nu}^{\alpha} p_{\alpha} + \frac{e}{m} F_{\nu\alpha} F^{\alpha\beta} p_{\beta}) + O(\tau_0^2)\,, \tag{5.2}$$

known as the Landau- Lifshitz (LL) equation. The LL equation has been solved for particular background fields such as constant electromagnetic fields [52, 53] and plane waves [54]. It would be interesting to investigate the role that symmetry plays in solving the LL equation and if all maximally super-integrable systems have an analytical solution for the LL equation.

Appendix A

Properties of Poisson brackets

The Poisson bracket of two functions $X(\mathbf{x}, \mathbf{p})$ and $Y(\mathbf{x}, \mathbf{p})$ on phase space is given by

$$\{X, Y\} = \sum_{i=1}^{n} \left(\frac{\partial X}{\partial x^i} \frac{\partial Y}{\partial p_i} - \frac{\partial X}{\partial p_i} \frac{\partial Y}{\partial x^i} \right) . \tag{A.1}$$

Given functions X, Y and Z on phase space and constants a and b the Poisson brackets obey the following properties:

Anti-symmetry:

$$\{X, Y\} = -\{Y, X\} . \tag{A.2}$$

Bilinearity:

$$\{X, aY + bZ\} = a\{X, Y\} + b\{X, Z\} . \tag{A.3}$$

Jacobi identity:

$$\{X,\{Y,Z\}\}+\{Y,\{Z,X\}\}+\{Z,\{X,Y\}\}=0\,. \tag{A.4}$$

Leibniz rule:

$$\{X,YZ\}=\{X,Y\}Z+Y\{X,Z\}\,. \tag{A.5}$$

Chain rule:

$$\{f(X),Y\}=f'(X)\{X,Y\}\,. \tag{A.6}$$

Bibliography

[1] The Nobel Prize in Physics 2018. https://www.nobelprize.org/prizes/physics/2018/summary/. Accessed: 2018-11-16.

[2] Donna Strickland and Gerard Mourou. Compression of amplified chirped optical pulses. *Optics Communications*, 55(6):447 – 449, 1985.

[3] Jörg Krüger and Wolfgang Kautek. Ultrashort pulse laser interaction with dielectrics and polymers. *Advances in Polymer Science*, 168:247–290, 2004.

[4] T. Tajima and G. Mourou. Zettawatt-exawatt lasers and their applications in ultrastrong-field physics. *Phys. Rev. ST Accel. Beams*, 5:031301, 2002.

[5] Vulcan 2020 Upgrade. https://www.clf.stfc.ac.uk/Pages/Vulcan-2020.aspx. Accessed: 2018-11-16.

[6] The ELI Project. https://eli-laser.eu/the-eli-project/. Accessed: 2018-11-16.

[7] The ELI Project. `https://www.eli-beams.eu/en/about/eli-infrastructure//`. Accessed: 2019-03-25.

[8] L. D. Landau and E. M. Lifshitz. *The Classical Theory of Fields (A Course of Theoretical Physics Vol. 2*. Butterworth-Heinemann Oxford, 1987.

[9] F. Gieres. About symmetries in physics. *arXiv:hep-th/9712154*.

[10] E. Noether. Invariante Variationsprobleme. *Gött. Nachr.*, 1918:235–257, 1918.

[11] H. Stephan. *Relativity*. Cambridge University Press, 2004.

[12] P. A. M. Dirac. Forms of relativistic dynamics. *Rev. Mod. Phys*, 21:392–399, 1949.

[13] H. Latal and W. Schweiger, editors. *Light-Cone Quantization: Foundations and Applications*, pages 55–142. Springer Berlin Heidelberg, 2001.

[14] O. Bühler. *A brief introduction to classical, statistical and quantum mechanics*. American Mathematical Society, 2006.

[15] S. Post, W. Miller Jr. and P. Winternitz. Classical and quantum superintegrability with applications. *J. Phys. A*, 46:423001, 2013.

[16] V. I Arnold. *Mathematical methods of classical mechanics*. New York: Springer- Verlag, 1978.

[17] S. Wojciechowski. Superintegrability of the Calogero-Moser system. *Physics Letters A*, 95:279 – 281, 1983.

[18] V. Turbiner P. Tempesta and P. Winternitz. Exact solvability of superintegrable systems. *J. Math. Phys*, 42:419–436, 2001.

[19] N. N. Nekhoroshev. Action-angle variables and their generalizations. *Trans. Moscow Math. Soc*, 26:180–198, 1972.

[20] W. Pauli. On the hydrogen spectrum from the standpoint of the new quantum mechanics. *Z. Phys.*, 36:336–363, 1926.

[21] V. Fock. On the theory of the hydrogen atoms. *Z. Phys.*, 98:145–154, 1935.

[22] J.Bertrand. Théorème relatif au mouvement d'un point attiré vers un centre fixe. *C.R.Acad.Sci.*, 77:849–853, 1873.

[23] R. P. Martinez-y-Romero H. N. Nunez-Yepez A. Gonzalez-Villanueva, E. Guillaumin-Espana and A. L. Salas-Brito. From circular paths to ellpitic orbits: a geometric approch to Keplers motion. *Eur. J. Phys.*, 19:431–438, 1998.

[24] J. Milnor. On the geometry of the Kepler problem. *Am. Math. Monthly*, 90:1123–1124, 1983.

[25] R. Jackiw. *Diverse topics in theoretical and mathematical physics*. Singapore: World Scientific, 1995.

[26] A. H. Taub. Orbits of charged particles in constant fields. *Phys. Rev.*, 73:786–798, 1948.

[27] T. Heinzl and A. Ilderton. Superintegrable relativistic systems in spacetime-dependent background fields. *J Phys. A*, 50, 2017.

[28] A. V. Arefiev A. P. L. Robinson and D. Neely. Generating "superpondero-motive" electrons due to a non-wake-field interaction between a laser pulse and a longitudinal electric field. *Phys. Rev. Letts.*, 111, 2013.

[29] *NIST Digital Library of Mathematical Functions*. http://dlmf.nist.gov/, Release 1.0.18 of 2018-03-27. F. W. J. Olver, A. B. Olde Daalhuis, D. W. Lozier, B. I. Schneider, R. F. Boisvert, C. W. Clark, B. R. Miller and B. V. Saunders, eds.

[30] I. S. Gradshteyn and I. M. Ryzhik. *Table of integrals, series, and products*. Academic Press, 5th revised edition edition, 1993.

[31] L. Šnobl, A. Marchesiello and P. Winternitz. Three-dimensional superintegrabile systems in a static electromagnetic field. *Journal of Physics A: Mathematical and Theoretical*, 48(39), 2015.

[32] G. Nordström. Zur Theorie der Gravitation vom Standpunkt des Relativitätsprinzips. *Annalen der Physik*, 42:533–554, 1913.

[33] A. Einstein and A. D. Fokker. Die Nordströmsche Gravitationstheorie vom Standpunkt des absoluten Differentialkalküls. *Annalen der Physik*, 44:321–328, 1914.

[34] Michio Kaku. *Quantum field theory: a modern introduction.* Oxford Univ. Press, 1993.

[35] P. W Higgs. Broken symmetries and the masses of gauge bosons. *Phys. Rev. Letters*, 13, 1964.

[36] ATLAS Collaboration. Observation of a new particle in the search for the Standard Model Higgs boson with the ATLAS detector at the LHC. *Physics Letters B*, 716(1), 2012.

[37] H. Yukawa. On the interaction of elementary particles. i. *Proceedings of the Physico-Mathematical Society of Japan. 3rd Series*, 17:48–57, 1935.

[38] D. H. Perkins. Nuclear disintegration by meson capture. *Nature*, 159, 1947.

[39] T. W. B. Kibble. Mutual refraction of electrons and photons. *Physical Review*, 150:1060–1069, 1966.

[40] T. Heinzl and A. Ilderton. A lorentz and gauge invariant measure of laser intensity. *Opt. Commun.*, 282:1879–1883, 2009.

[41] Lowell S Brown and TWB Kibble. Interaction of intense laser beams with electrons. *Physical Review*, 133(3A):A705, 1964.

[42] Joseph H Eberly and Howard R Reiss. Electron self-energy in intense plane-wave field. *Physical Review*, 145(4):1035, 1966.

[43] A. Ghose-Choudhury and P. Guha. Superintegrable systems with position dependent mass: master symmetry and action-angle methods. *arXiv:1804.01074 [nlin.SI]*, 2018.

[44] L. Ansell, T. Heinzl and A. Ilderton. Superintegrable relativistic systems in scalar background fields. *Journal of Physics A: Mathematical and Theoretical*, 51(49):495203, 2018.

[45] N. Straumann. Reflections on gravity. *arXiv:astro-ph/0006423*, 2000.

[46] J.Struckmeier and C. Riedel. Canonical transformations and exact invariants for timedependent Hamiltonian system. *Annalen der Physik*, 11:15–38, 2002.

[47] L. Mason, T. Adamo, E. Casali and S. Nekovar. Scattering on plane waves and the double copy. *Class. Quant. Grav.*, 35, 2018.

[48] O. Klein. Die Reflexion von Elektronen an einem Potentialsprung nach der relativischen Dynamik von Dirac. *Z. Phys.*, 53(853-868), 1929.

[49] M. Abraham. *Theorie der Elektizitat.* Leipzig, 1905.

[50] H.A. Lorentz. *The Theory of Electrons and Its Applications to the Phenomena of Light and Radiant Heat.* Leipzig, New York, 1909.

[51] P. A. M. Dirac. Classical theory of radiating electrons. *Proc. Roy. Soc. A*, 167, 1938.

[52] Y. Yaremko. Exact solution to the landau-lifshitz equation in a constant electromagnetic field. *Journal of Mathematical Physics*, 54, 2013.

[53] G. Ares de Parga and R. Mares. Exact solution of the herrera equation of motion in classical electrodynamics. *J. Math. Phys.*, 40, 1999.

[54] A. Di Piazza. Exact solution of the landau-lifshitz equation in a plane wave. *Letters in Mathematical Physics*, 83:305–313, 2008.